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Study of Linear Feedback Systems with Periodic Parameters Through an Extension of the Floquet Theory

by
Imsong Lee

December 1962

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SYSTEMS THEORY LABORATORY

STANFORD ELECTRONICS LABORATORIES

STANFORD UNIVERSITY · STANFORD, CALIFORNIA

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ABSTRACT

The basic properties of multi-variable linear feedback systems with periodically varying parameters are investigated. This class of systems is described by linear differential equations with periodic coefficients in the "state space." The classical Floquet theory on linear differential equations with continuous, periodic coefficients has been extended to treat linear differential equations with piece-wise continuous, periodic coefficients. The extended Floquet theory is applied to the stability analysis of modulated feedback control systems with continuous and piece-wise continuous carriers.

It is shown that analysis and synthesis of many classes of linear feedback systems may be formulated from a unified point of view by using Volterra integral equations of the second kind.

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I. INTRODUCTION

A. BACKGROUND OF THIS WORK

Today a vast number of communication and control systems are carrier frequency systems at least in part; that is, they contain carrier-frequency links which typically consist of modulators (multipliers), A.C. amplifiers, and demodulators (multipliers). The product-type modulator, which simply multiplies one time function called the input signal by a certain periodic time function called the carrier, may be treated as a periodically time-varying gain or as a parameter. Symbolically, this type of circuit is represented as shown in Fig. 1, where $m(t)$ is the product-type modulator.

In many communication and control systems the frequency spectra of the information-bearing signals are several orders of magnitude below the carrier frequency; i.e., the ratio of the carrier frequency to the highest input signal frequency is very high. Under this condition, the carrier-frequency link shown in Fig. 2 can be approximated by an equivalent time-invariant linear network [Ref. 1, pp. 60-68]. By experience, such an approximation has proven to be adequate for the analysis and design of carrier-frequency systems which satisfy the condition stated above. In this case, a linear system with periodic parameters is reduced to an equivalent stationary linear system, so

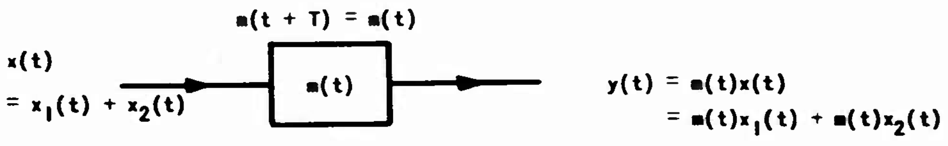


FIG. 1. THE PRODUCT-TYPE MODULATOR (MULTIPLIER).

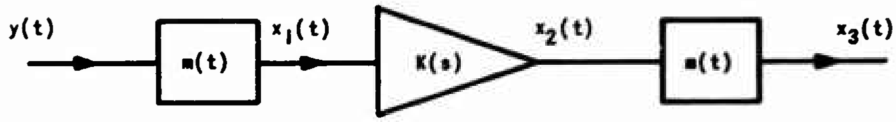


FIG. 2. A TYPICAL CARRIER-FREQUENCY LINK.

that one can make use of the frequency domain methods developed for stationary linear systems.

Such a simplification cannot be obtained, however, in cases where the external input is changing at a rate comparable with the carrier frequency--that is to say, when the upper limit of the frequency range of the input signals is comparable with the carrier frequency. For example, consider the design of a carrier-type D.C. feedback amplifier with a high performance index (a figure of merit). The performance index of interest in this case is the largest gain-bandwidth product obtainable with specified solid-state electronic components. Assume also that the carrier is supplied by the photo-conductor chopper and the upper limit of its fundamental frequency is fixed by the current state of technology. Under this condition, the upper limit of the signal frequency range becomes comparable with the carrier frequency because of the wide bandwidth requirement.

The carrier frequency system is a special case of a linear system in which one or more system parameters vary periodically with time. The well-known modulated control systems such as carrier servo systems, chopper modulated D.C. feedback amplifiers, and sampled-data systems are typical modern engineering examples of linear systems with periodic parameters.

Not only the modulated control systems and modern communication systems using sinusoidal or pulse-amplitude modulations, but also

oscillations of locomotive mechanisms [Ref. 2], two-dimensional motion of a pendulum suspended on a spring [Ref. 3], coupled pendulums with oscillating supports [Ref. 4], and perturbed motion of two spring-coupled masses [Ref. 5] in an orbiting space satellite provide physical examples of linear systems with periodically varying parameters. Such physical systems are called in this study linear systems with periodic parameters, and are represented mathematically by linear differential equations with periodic coefficients.

B. STATEMENT OF THE PROBLEM AND ITS FORMULATION

In order to find a comprehensive method applicable to the analysis and, hopefully, applicable to the design of modulated control systems in general, it is desirable to treat the modulated control systems as a special class of periodic linear systems. Then the problem is to seek a proper mathematical representation of periodic linear systems from which we can readily derive useful methods for the analysis and design of the modulated control systems. The central purpose of this work is the development of simple methods for the analysis and design of periodic linear systems. In order to achieve this purpose it is necessary to choose an appropriate mathematical representation for the system under study.

The representation of a periodic linear system by a linear differential equation with periodic coefficients in "state space" provides the general formulation that is sought above. The state space is an abstract space defined by the "state variables" of a dynamic system. The number of state variables is equal to the number of variables necessary and sufficient to describe the "future state" of a given dynamic system uniquely, if the present values of the variables and the external input are known. There are many ways of defining the state variables, but any set of variables having the properties just mentioned are called "state variables."

The theoretical as well as practical advantages which may come from this general representation of a linear system with periodic parameters have not been clearly understood and realized except in the special case of periodically sampled systems [Refs. 6 and 7]. The differential

equation formulation in state space allows us to take full advantage of the well-known Floquet theory in the stability analysis of a linear system with periodic coefficients. One can obtain from the Floquet theory not only valuable insight into the fundamental nature of linear systems with periodic coefficients, such as the form of natural response and asymptotic behavior, but also a computational tool for calculation of the characteristic roots.

An alternative approach to the problem of system representation has also been studied. It is shown in this study that general linear feedback systems can be represented by Volterra integral equations of the second kind. Well-known stationary and sampled-data feedback systems are described by special cases of Volterra integral equations of the second kind.

C. BRIEF SURVEY OF PREVIOUS WORK

Historically, the linear differential equations with periodic coefficients were first studied in mechanics and astronomy. Two of the famous early examples are the Mathieu equation and the Hill equation [Ref. 8].

The most definitive work in the theory of linear differential equations with periodic coefficients appeared in the celebrated paper by the French mathematician M. G. Floquet, published in 1883 [Ref. 9]. The famous memoir by the gifted Russian mathematician A. M. Lyapunov concerning stability of motion was published in 1892, nine years after the publication of Floquet's paper [Ref. 10]. Lyapunov's memoir deals with very general problems concerning stability of linear and nonlinear dynamic systems. Though he followed a different path, Lyapunov arrived independently at the same conclusion as Floquet on the stability of linear differential equations with continuous, periodic coefficients.

The work of Floquet and Lyapunov made fundamental contributions to an understanding of the "natural behavior" and stability of linear systems with periodic coefficients. Although the theories developed by the two mathematicians were simple and straightforward, their applications have been mainly limited to second order systems [Ref. 11; also Ref. 12, pp. 59-66].

It is not necessary, nor is it even desirable, to make use of Floquet theory in the analysis and design of conventional carrier frequency systems, such as A.M. (amplitude modulation) communication systems, where stability problems are nonexistent because of the particular nature of the physical systems involved. In these cases, the sinusoidal steady-state responses are obtained simply by use of the Fourier transform method.

One can also use the simple frequency domain technique in the stability analysis of special classes of modulated control systems [Refs. 2 and 13]. However, a complete theory applicable to the stability analysis of general modulated control systems has not hitherto been developed.

In order to develop such a general theory, it is necessary to extend the original Floquet theory to linear differential equations with piece-wise continuous, periodic coefficients. Many modern carrier frequency systems use not only continuous sinusoidal carriers but also discontinuous, pulse-train carriers and thus give rise to a class of systems represented by linear differential equations with piece-wise continuous periodic coefficients.

The Hill-Meissner equation [Ref. 2] is an example of a second-order linear differential equation with piece-wise continuous, periodic coefficients. L. A. Pipes [Ref. 13] presented a matrix solution of this type of equation without formulating a general theory applicable to equations of order higher than the second.

Traditionally the ~~work of Russian mathematicians~~ on the stability theory of ordinary differential equations has been very extensive. The contributions of the Russian school to the theory of linear differential equations with periodic coefficients up to 1956 appears to be summarized in a paper by V. M. Starzhinskii [Ref. 14]. Since 1956 Aizerman and Gantmacher, two prominent applied mathematicians in the Soviet Union, have made significant contributions on the stability (in the Lyapunov sense) of periodic solutions of a nonautonomous differential equation [Refs. 15 and 16]. Although the part of their work concerning the properties of the zero solution of a linear approximation contains some concepts related to the stability of linear systems with periodic

coefficients, they have not investigated explicitly the necessary and sufficient conditions for the stability of linear systems with piece-wise continuous, periodic coefficients.

Since the development of the theory of the integral equation by the Italian mathematician Vito Volterra in 1886 (see the biography by E. T. Whittaker in Ref. 17, p. 11), this branch of mathematics found increasing applications in physics [Ref. 18], in engineering [Refs. 19 and 20], and in the theory of differential equations [Ref. 21; also Ref. 12, p. 55].

The representation of a stationary linear feedback system by a special type of Volterra integral equation of the second kind has recently been mentioned [Ref. 22; also Ref. 12, p. 29]. However, the theoretical and practical advantages that may be obtained from the representation of general linear feedback systems by Volterra integral equations of the second kind have not been investigated.

D. CONTRIBUTIONS OF THIS WORK

The classical theory of Floquet on the linear differential equations with continuous, periodic coefficients cannot be applied to the stability analysis of modulated control systems having discontinuous pulse-train carriers. The reason is that such systems are represented by linear differential equations with piece-wise continuous periodic coefficients. A proper extension of Floquet theory to linear systems with piece-wise continuous periodic coefficients is considered as the first contribution of this dissertation.

The second contribution is a clear physical interpretation of the Floquet theory and its applications to the analysis of, and, to a limited degree, the design of, modulated control systems. A practical computational method for calculation of the characteristic exponents and for determination of stability margins is considered particularly significant for engineering applications.

The third contribution is the representation of general linear feedback systems by Volterra integral equations of the second kind. The transfer functions of stationary linear feedback systems and sampled-data control systems are derived as special cases of this representation.

Although the full potentialities of this method have not been investigated exhaustively, it appears that this representation may offer very valuable insight into the basic properties of linear feedback systems.

II. EXTENSION AND INTERPRETATION OF FLOQUET THEORY

A. INTRODUCTION

1. State Vector Representation of Dynamic Systems

A dynamic system may be described in many different ways. Throughout this chapter we represent a dynamic system by a system of first order differential equations in n-dimensional "state space."

$$\dot{x}_i(t) = f_i(x_1, \ldots x_n, t) + b_i(t) \; ; \quad t \geq 0 \, , \quad i = 1, 2, \ldots n \tag{2.1}$$

In order to simplify the subsequent discussions, this system of simultaneous differential equations is replaced by a more compact vector differential equation.

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}, t) + \underline{b}(t) \; ; \quad t \geq 0 \, , \tag{2.2}$$

where $\underline{x}$, $\underline{b}$, and $\underline{f}$ are $n \times 1$ column vectors.

The force-free linear dynamic system is normally represented by the following equation:

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) \; ; \quad t \geq 0 \, , \tag{2.3}$$

where $A(t)$ is an $n \times n$ matrix.
The state vector $\underline{x}(t)$ is defined as a set of state variables which determine the future state of a dynamic system uniquely if their present values are given. For instance, if the state at $t = t_o$, $\underline{x}(t_o)$, is known, then the state vector at any future time, $\underline{x}(t)$ for $t > t_o$, is uniquely determined by differential equations such as Eq. (2.2) or (2.3).

There seems to be no unique way of choosing state variables of a given dynamic system. Such variables could be the canonical coordinates and momenta as in the Hamiltonian formulation of classical mechanics [Ref. 23], or they could be some variables and their derivatives

observable in a given system [Ref. 24].

2. Fundamental Matrix and Impulse Response

First we consider a stationary linear system in order to calculate explicitly the fundamental matrix and understand clearly its relation to the familiar concept of impulse response.

$$\underline{\dot{x}}(t) = A\underline{x}(t) + \underline{b}(t) \ ; \quad t_o \le t < \infty \ , \tag{2.4}$$

where A is a constant $n \times n$ matrix and $\underline{b}(t)$ is a bounded $n \times 1$ column matrix.

The solution of this equation is [Ref. 25, p. 11]:

$$\underline{x}(t) = \Phi(t - t_o)\underline{x}(t_o) + \int_{t_o}^{t} \Phi(t - \tau)\ \underline{b}(\tau)\ d\tau \ , \tag{2.5}$$

where $\Phi(t) = \exp At$.

The second term on the right-hand side of the above equation is a matrix convolution integral where the $n \times n$ nonsingular matrix $\Phi(t)$ is called a fundamental matrix or a fundamental system of solutions.

If the external input $\underline{b}(t)$ is absent, we have

$$\underline{x}(t_1) = \Phi(t_1 - t_o)\underline{x}(t_o) \ ,$$

and $\Phi(t_1 - t_o)$ is a transformation matrix which describes the transition of "state" from t_o to t_1. Because of this property, $\Phi(t)$ is also called the <u>state transition matrix</u>.

It is very instructive to examine the properties of the fundamental matrix $\Phi(t)$ for a general linear system.

$$\underline{\dot{x}} = A(t)\underline{x} \ , \quad 0 \le t_o \le t < \infty \ , \tag{2.6}$$

where $A(t)$ is a time-dependent, bounded and continuous $n \times n$ matrix. The solution may be formally written as

$$\underline{x}(t) = \Phi(t, t_o)\underline{x}(t_o) , \qquad \Phi(t_o, t_o) = I . \tag{2.7}$$

In this case the fundamental matrix $\Phi(t, t_o)$ depends on the time of excitation t_o as well as the time interval $t - t_o$. This is exactly analogous to a time-variable impulse response $h(t, \tau)$ [Ref. 26]. For a stationary linear system, we have $\Phi(t, t_o) = \Phi(t - t_o)$. We define the fundamental matrix $\Phi(t, t_o)$ and its properties as follows:

1. The fundamental matrix satisfies the homogeneous matrix differential equation associated with the vector equation (2.6):

$$\left.\begin{aligned} &\dot{\Phi}(t, t_o) = A(t)\Phi(t, t_o) , \quad 0 \le t_o \le t < \infty , \\ &\Phi(t_o, t_o) = I . \end{aligned}\right\} \tag{2.8}$$

2. Each column of the fundamental matrix $\Phi(t, t_o)$ is a linearly independent solution of the vector differential equation (2.6).

$$\begin{gathered} \dot{\underline{\phi}}_i = A(t)\, \underline{\phi}_i , \quad 0 \le t_o \le t < \infty , \\ i = 1, 2, \ldots n ; \quad \underline{\phi}_i = \underline{\phi}_i(t, t_o) \\ \phi_{ij}(t_o, t_o) = \delta_{ij} (\text{Kronecker delta}), \quad j = 1, 2, \ldots n . \end{gathered} \tag{2.9}$$

3. The determinant of $\Phi(t, t_o)$ is given by the Jacobi-Liouville formula [Ref. 12, p. 19]. It is seen from this that $\Phi(t, t_o)$ is never singular.

$$\det \Phi(t, t_o) = \exp \int_{t_o}^{t} \mathrm{Tr}\, A(\tau)\, d\tau , \tag{2.10}$$

where $\mathrm{Tr}\, A(\tau) = \sum_{i=1}^{n} a_{ii}(\tau)$.

Since it is often convenient, however, to choose $t_o = 0$, we shall use the following notation for this case.

$$\Phi(t, 0) \equiv \Phi(t) ; \qquad \Phi(0, 0) \equiv \Phi(0) = I . \tag{2.11}$$

Now it is possible to express $\Phi(t, t_o)$ in terms of $\Phi(t)$ by noting that $\Phi(t)$ is the fundamental matrix for the vector equation

$$\dot{\underline{x}} = A(t)\underline{x}\ , \quad \underline{x}(0) \neq \underline{0}\ , \quad 0 \leq t < \infty\ . \tag{2.12}$$

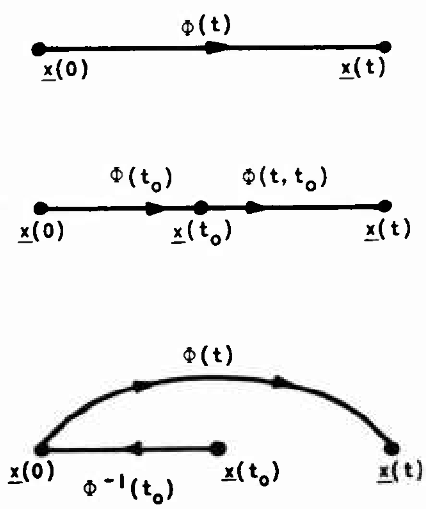
This is the same equation as (2.6), only with a different interval and initial conditions. The solution is

$$\underline{x}(t) = \Phi(t)\underline{x}(0)\ . \quad \Phi(0) = I\ . \tag{2.13}$$

From the group of equations, (2.6) to (2.13), we deduce

$$\Phi(t, t_o) = \Phi(t)\ \Phi^{-1}(t_o)\ . \tag{2.14}$$

A graphical interpretation of the above equation is illuminating and helpful in a discussion of the Floquet theory. From Eqs. (2.7) and (2.13), one may regard the fundamental matrix as a linear operator representing the transfer (or transmission) characteristics of a system described by differential equations (2.6) and (2.13). This can be illustrated graphically as shown in Fig. 3.



$$\underline{x}(t) = \Phi(t)\underline{x}(0)\ .$$

$$\underline{x}(t_o) = \Phi(t_o)\underline{x}(0)\ ,$$
$$\underline{x}(t) = \Phi(t, t_o)\underline{x}(t_o)\ .$$

$$\begin{aligned} \underline{x}(t) &= \Phi(t)\Phi^{-1}(t_o)\underline{x}(t_o) \\ &= \Phi(t, t_o)\underline{x}(t_o)\ , \\ \Phi(t, t_o) &= \Phi(t)\Phi^{-1}(t_o)\ . \end{aligned}$$

FIG. 3 A GRAPHICAL REPRESENTATION OF THE FUNDAMENTAL MATRIX.

A nonstationary linear system with an external input $\underline{b}(t)$ is represented by

$$\dot{\underline{x}} = A(t)\underline{x} + \underline{b}(t) , \quad t_o \leq t < \infty , \tag{2.15}$$

where $\underline{b}(t)$ is an $n \times 1$ vector.
The solution may be formally written as:

$$\underline{x}(t) = \Phi(t)\Phi^{-1}(t_o)\underline{x}(t_o) + \int_{t_o}^{t} \Phi(t)\Phi^{-1}(\tau)\underline{b}(\tau)\, d\tau \tag{2.16}$$

and this reduces to a convolution integral if $\Phi(t) = \exp At$. The matrix kernel $\Phi(t)\Phi^{-1}(\tau)$ may be regarded as a matrix form of a time-variable impulse response $h(t, \tau)$.

Having found a general form for a forced response of the nonstationary linear system, Eq. (2.15), the next important question is the boundedness of response.

3. Definition of Stability

The physical idea behind the mathematical concept of stability is closely related to a "bounded" response of a physical system when it is driven by an external input. Usually a mathematical definition of stability is based on the boundedness (i.e., the property of having specific bounds or limits) of solutions of differential equations. This gives rise to many definitions of stability, because it is possible to define the boundedness of solutions of differential equations in many different ways [Ref. 27, Chapter 4].

For the purpose of the type of physical applications we have in mind, however, the definition of stability based on an input-output relationship seems to be adequate. We therefore adopt the following definition of stability for the subsequent discussions [Ref. 28]: A system is said to be stable if, and only if, its output is bounded for a bounded input.

In order to put this definition of stability into a more precise mathematical form, it is necessary to define the "magnitude" of a vector $\underline{x}(t)$ in some convenient way. The magnitude of a scalar $y(t)$

may be defined by its absolute value, $|y(t)|$. The "magnitude" of a vector is called a norm and it may be defined in several ways [Ref. 29, pp. 71-72].

For purposes of this paper a norm of a vector $\underline{x}(t)$ is defined to be:

$$||\underline{x}(t)|| = \sup_i \left\{ |x_i(t)| \right\}, \quad i = 1, 2, \ldots n . \tag{2.17}$$

By definition, the forced response of a linear system as shown by Eq. (2.16) is stable, if and only if,

$$||\underline{x}(t)|| < M \quad \text{for all} \quad t \geq t_o , \tag{2.18}$$

given that $||\underline{b}(t)|| < M_2$ for all $t \geq t_o$,
where M_1 and M_2 are finite, positive, real numbers.

In the case of a single-variable linear system which is characterized by the input-output relationship

$$x(t) = \int_o^\infty h(t, \tau)\, f(\tau)\, d\tau ,$$

it is well-known that the necessary and sufficient condition for stability [Ref. 26] is given by

$$\int_o^\infty |h(t, \tau)| d\tau < M \quad \text{for all} \quad t \geq 0 ,$$

provided that we exclude "impulse" functions and the higher derivatives of the impulse functions.

We can state the exactly analogous condition for multi-variable linear systems [Ref. 30]: The output of a linear system as given by Eq. (2.16) is bounded for a bounded input if, and only if,

$$\int_{t_o}^t ||\Phi(t)\Phi^{-1}(\tau)|| d\tau < M \quad \text{for all} \quad t \geq t_o , \tag{2.19}$$

where M is a finite, positive, real number.

We shall refer to this statement as the Fundamental Theorem on Stability (of linear systems).

B. EXTENSION OF FLOQUET THEORY

1. Introduction

The linear system in which one or more system parameters may vary periodically with time is represented by a linear differential equation with periodic coefficients:

$$\dot{\underline{x}} = A(t)\underline{x} , \quad 0 \leq t < \infty , \qquad (2.20)$$
$$A(t + 1) = A(t) ,$$

where the elements of the matrix $A(t)$ are continuous functions of time $a_{ij}(t)$ of normalized period 1. The period has been normalized to 1 as a matter of convenience. This system can be dealt with adequately by the classical theory of Floquet.

If some of the elements $a_{ij}(t)$ have a finite number of discontinuities within the period $0 \leq t \leq 1$, then it is necessary to extend the classical theory to deal with such piece-wise continuous cases. First we consider a simple introductory example, in order to be better prepared for a discussion of the general theory.

Example 1: A simple linear system with periodic coefficients

Let us consider the following first-order linear differential equation with periodic coefficients:

$$\dot{x} + a(t)x = 0 , \quad x(0) \neq 0 , \quad 0 \leq t < \infty , \qquad (2.21)$$
$$a(t + 1) = a(t)$$

First of all, it will be shown that a solution of this equation is not necessarily a periodic function of time $x(t)$ of period 1. To make this example more concrete, we may regard the above equation as representing the feedback system shown in Fig. 4.
The system equation is

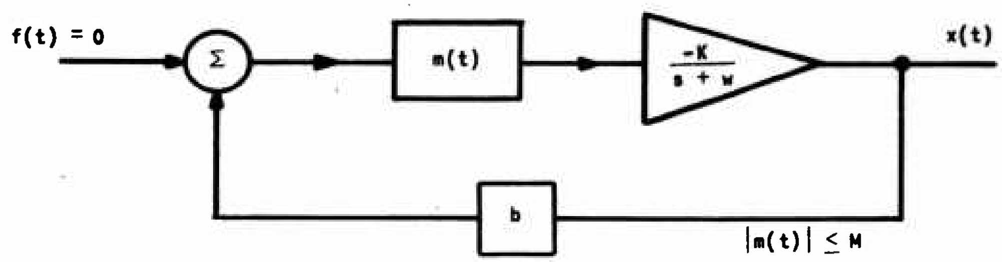


FIG. 4. A SIMPLE LINEAR SYSTEM WITH A PERIODIC PARAMETER.

$$\dot{x} + [w + Kbm(t)]x = 0 , \quad x(0) \neq 0 ,$$

where $m(t + 1) = m(t)$ and $a(t) = w + Kbm(t)$.
The solution is obtained by direct integration:

$$x(t) = x(0) \exp[-\int_0^t a(\tau)\, d\tau] = x(0)\, \phi(t) ,$$

$$x(t + 1) = x(0) \exp[-\int_0^{t+1} a(\tau)\, d\tau] = \exp[-\int_0^t a(\tau)\, d\tau] x(t)$$

$$= cx(t) .$$

It is clear from the above equations that $x(t)$ is not a periodic solution unless $\int_0^1 a(\tau)\, d\tau = 0$.

We note the following important properties of the solution:

1. $\phi(t) = \exp[-\int_0^t a(\tau)\, d\tau]$ is a 1×1 fundamental matrix. It is the solution of

$$\dot{\phi} + a(t)\phi = 0 , \quad \phi(0) = 1 , \quad 0 \leq t < \infty .$$

2. $$\phi(t + 1) = c\phi(t) ,$$

   where $c = \phi(1) = \exp[-\int_0^1 a(\tau)\, d\tau]$.

3. From the two equations shown above,

$$\phi(t + 1) = c\phi(t) \ , \quad \text{and}$$
$$x(t + 1) = cx(t) \ ,$$

we deduce the following difference equations:

$$\phi(k + 1) = c\phi(k) \ ,$$
$$x(k + 1) = cx(k) \ .$$

Both the fundamental matrix $\phi(t)$ and the solution $x(t)$ satisy the same difference equation.

4. The necessary and sufficient condition for stability of the solution $x(t)$ is obtained readily from 2. and 3. as follows: The linear periodic system (Eq. 2.21) is stable if, and only if,

$$|c| = \exp[-\int_0^1 a(\tau)\, d\tau] < 1 \ .$$

The generalization of the above results in terms of an n-dimensional system for the case in which $a(t)$ is continuous leads directly to the classical Floquet theory.

2. The Classical Theory of Floquet on Linear Systems with Continuous, Periodic Coefficients

We start from the n-dimensional vector differential equation (2.20):

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) \ , \quad \underline{x}(0) \neq \underline{0} \ , \quad 0 \leq t < \infty \ ,$$
$$A(t + 1) = A(t) \ ; \quad n \times n \ \text{matrix} \ ,$$

where the elements of $A(t)$ are continuous, periodic functions $a_{ij}(t)$ of period 1. The formal solution is

$$\underline{x}(t) = \Phi(t)\underline{x}(0) \ , \tag{2.22}$$

and we examine the basic properties of the fundamental matrix $\Phi(t)$ which were investigated by Floquet and Lyapunov. The following properties are noted:

1. Both $\Phi(t)$ and $\Phi(t + 1)$ satisfy the homogeneous matrix differential equation associated with Eq. (2.20). This can be proved

simply by noting that both $\underline{x}(t) = \Phi(t)\underline{x}(0)$ and $\underline{x}(t + 1) = \Phi(t + 1)\underline{x}(0)$ are the solutions of Eq. (2.20), and consequently they must satisy the same equation. By substituting $\underline{x}(t)$ and $\underline{x}(t + 1)$ into Eq. (2.20) we obtain:

$$\dot{\Phi}(t) = A(t)\,\Phi(t)$$

and

$$\dot{\Phi}(t + 1) = A(t + 1)\,\Phi(t + 1) = A(t)\,\Phi(t + 1) .$$

2. Since each column of the two matrices $\Phi(t)$ and $\Phi(t + 1)$ satisfies the differential equation (2.20), and there are not more than n independent solutions, the columns of $\Phi(t + 1)$ must be linear combinations of those of $\Phi(t)$. This relationship may be stated formally as

$$\Phi(t + 1) = \Phi(t)C, \quad 0 \leq t < \infty , \tag{2.23}$$

where C is a constant, nonsingular $n \times n$ matrix because $C = \Phi(1)$ from the above equation.

3. From the group of equations shown above, Eqs. (2.20 to 2.23) deduce the all-important linear difference equations. This is a very critical step, because we reduce a linear differential equation with periodic coefficients, Eq. (2.20), to a linear difference equation with <u>constant coefficients</u>.

   (a) From Eq. (2.23) we have a matrix difference equation

$$\Phi(k + 1) = \Phi(k)C = C\Phi(k) , \tag{2.24}$$

   $k = 0, 1, 2, \ldots$
   From Eqs. (2.22), (2.23) and (2.24) we obtain a vector difference equation

$$\underline{x}(k + 1) = C\underline{x}(k) , \tag{2.25}$$

   $k = 0, 1, 2, \ldots$
   where $C = \Phi(1)$ is called the discrete transition matrix.

   (b) The solutions of the above two difference equations are

$$\Phi(k) = \Phi(0)C^k = C^k , \tag{2.24a}$$

$$\underline{x}(k) = C^k\underline{x}(0) , \tag{2.25a}$$

and it can be easily verified by direct substitution that $\Phi(k)$ and $\underline{x}(k)$ satisfy Eqs. (2.24) and (2.25) respectively.

(c) The determinant of the transition matrix C is obtained from the Jacobi-Liouville formula as previously given by Eq. (2.10) [Ref. 12, p. 57].

$$\det C = \det \Phi(1) = \exp \int_{0}^{1} \mathrm{Tr}\, A(\tau)\, d\tau = z_1 z_2 \ldots z_n \,, \qquad (2.26)$$

where z_i are the eigenvalues of the matrix C and are called the characteristic roots of Eq. (2.20).

4. The periodic transition of the state vector $\underline{x}(t)$ between two successive periods from $t = k$ to $t = k + 1$ is described by the difference equation (2.25), but this only describes the periodic transition starting from $t = 0$. If we want to describe the periodic transition of the state vector starting from some arbitrary initial time, $t \neq 0$, it is necessary to derive a new difference equation.

(a) The continuous transition of state starting from $t \geq 0$ is described by

$$\dot{\underline{y}}(t) = A(t)\underline{y}(t) \,, \quad t_o \leq t \leq \infty \,, \qquad (2.27)$$

$$A(t + 1) = A(t) \,, \quad \underline{y}(t_o) = \underline{x}(t_o) \,,$$

$$\underline{y}(t) = \Phi(t, t_o)\underline{y}(t_o) = \Phi(t)\Phi^{-1}(t_o)\underline{y}(t_o) \,. \qquad (2.28)$$

From Eq. (2.14) and (2.23) we have

$$\Phi(t + 1, t_o) = \Phi(t, t_o)C' \,, \qquad (2.29)$$

$$C' = \Phi(t_o)C\,\Phi^{-1}(t_o) \,, \qquad (2.30)$$

and from the above equations we obtain a new difference equation

$$\underline{y}(i + 1) = C'\underline{y}(i)$$
$$i = k + t_o, \quad k = 0, 1, 2, \ldots \qquad (2.31)$$

(b) Because of the periodicity of coefficient matrix $A(t)$, we find the periodic transition characteristics represented by

$$\Phi(\tau + k + 1, k + 1) = \Phi(\tau + k, k) = \Phi(\tau)$$
$$0 \leq \tau \leq 1, \quad k = 0, 1, 2, \ldots \qquad (2.32)$$

The significance of this equation may be best illustrated by graphical representation (Fig. 5).

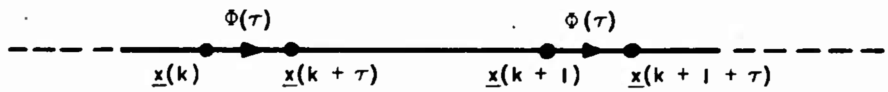


FIG. 5. A PERIODIC TRANSITION CHARACTERISTIC.

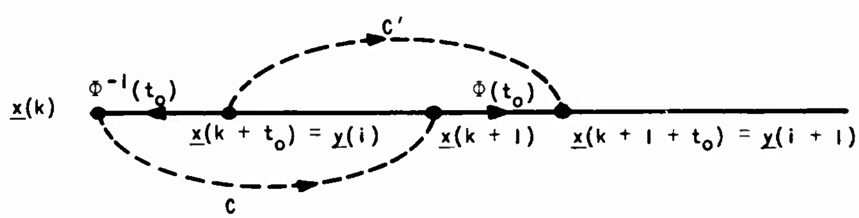
Fig. 5 illustrates the relation:

$$\underline{x}(k + \tau) = \Phi(k + \tau, k)\underline{x}(k) = \Phi(\tau)\underline{x}(k) , \tag{2.33}$$

$$0 \leq \tau \leq 1, \quad k = 0, 1, 2, 3, \ldots$$

This equation shows very clearly that we know the transition characteristics of the state vector $\underline{x}(t)$ for all time, if it is known for any one period. Thus, stated in formal language, the solution of the linear differential equation (2.20) is known for all time if it is known for any one period.

(c) It is also instructive to derive Eqs. (2.30) and (2.31) from a graphical representation (Fig. 6).



$$\underline{y}(i + 1) = \Phi(t_o)C\,\Phi^{-1}(t_o)\underline{y}(i)$$
$$= C'\,\underline{y}(i) .$$

FIG. 6. A GRAPHICAL REPRESENTATION OF THE DIFFERENCE EQUATION.

5. The analytic form of a fundamental matrix $\Phi(t)$ is given in Ref. 25, p. 28, as

$$\Phi(t) = \Pi(t) \exp Bt , \tag{2.34}$$

$$\Pi(t + 1) = \Pi(t) , \quad \Pi(0) = 1 .$$

We can deduce a number of important results from the above equation:

(a) The exponential matrix $\exp Bt$ and the transition matrix C are related by the equation

$$C = \exp B . \tag{2.35}$$

The eigenvalues of the matrix B are called the characteristic exponents of Eq. (2.20) and are obtained from the equation $\det[\beta I - B] = 0$. The characteristic exponents β_i are related to the characteristic roots z_i by a logarithmic equation

$$z_i = e^{\beta_i + j2k\pi} \tag{2.36}$$

or

$$\beta_i = \ln z_i = \ln |z_i| + j(\theta_i + 2k\pi) , \tag{2.37}$$

$$k = 0, \pm 1, \pm 2, \ldots ; \quad i = 1, 2, \ldots n .$$

We shall take only the principal values, unless stated otherwise.

$$\beta_i = \ln |z_i| + j\theta_i . \tag{2.38}$$

The above relations between the characteristic exponents and the characteristic roots are valid not only for the case in which all the eigenvalues are distinct, but also for the case of multiple eigenvalues. This may be proved simply by transforming the two matrices B and C into Jordan normal form [Ref. 31, pp. 67-120].

(b) If the matrix B is of simple structure [Ref. 32], one can obtain a new fundamental matrix $\Psi(t)$ by a similarity transformation which diagonalizes matrix B.

$$\begin{aligned}\Psi(t) &= T^{-1}\Phi(t)T = [T^{-1}\Pi(t)T][T^{-1}e^{Bt}T] \\ &= P(t) \exp B_d t \end{aligned} \tag{2.39}$$

where B_d is a diagonal matrix, and T is a constant matrix. $\Psi(t)$ is a new fundamental matrix associated with Eq.(2.20)

if we introduce the change of variable, $\underline{x} = T\underline{y}$. The analytic form of each column is deduced from Eq. (2.39).

$$\underline{\psi}_i(t) = \underline{p}_i(t)e^{\beta_i t} , \qquad (2.40)$$

$$i = 1, 2, \ldots n ,$$

where $\underline{\psi}_i(t)$ and $\underline{p}_i(t)$ represent the i th column of the matrices $\Psi(t)$ and $P(t)$.
Any solution $\underline{x}(t) = T\underline{y}(t)$ of the differential equation (2.20) may be considered as a vector in the n-dimensional space spanned by the set of vectors $T\underline{\psi}_i(t)$. Furthermore, this set of vectors $\underline{\psi}_i(t)$ satisfy the difference equation:

$$\underline{\psi}_i(t + 1) = z_i \, \underline{\psi}_i(t) , \qquad (2.41)$$

$$i = 1, 2, \ldots n ,$$

where $z_i = e^{\beta_i}$.

(c) Physically, the set of vectors $T\underline{p}_i(t)e^{\beta_i t}$; $i = 1, 2, \ldots n$, represents the normal modes of the system represented by Eq. (2.20). Consequently any force-free behavior of the system may be resolved into each normal mode. Since $\underline{p}_i(t + 1) = \underline{p}_i(t)$, one can expand each component of $\underline{p}_i(t)$ in a Fourier series; or, for that matter, the fundamental matrix $\Psi(t) = P(t) \exp B_d t$ can be expanded in a Fourier series, because $P(t) = P(t + 1)$. A typical component of the fundamental matrix will be of the form

$$\psi_{ki}(t) = e^{\beta_i t} \sum_{m = -\infty}^{\infty} a_{km} e^{jm\Omega t} ,$$

where $\Omega = 2\pi/T = 2\pi$ (carrier frequency) .
The terms $e^{jm\Omega t}$ for large integer values of m, give rise to high frequency ripples normally present in any carrier frequency system [Ref. 1, pp. 60-67].

(d) If the matrix B has repeated eigenvalues and is not of simple structure, then it is always possible to put B into a Jordan normal form. The fundamental matrix $\Psi(t)$ is then of the form

$$\Psi(t) = Q(t) \exp Jt , \qquad (2.42)$$

$$\text{where} \quad J = \begin{bmatrix} \beta_1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \beta_1 & 1 & & & & & & & & \\ 0 & 0 & \beta_1 & & & & & & & & \\ \cdot & & & \cdot & & & & & & & \\ \cdot & & & & \cdot & & & & & & \\ \cdot & & & & & \cdot & 0 & 0 & & & \\ \cdot & & & & & & \beta_2 & 1 & & & \\ \cdot & & & & & & 0 & \beta_2 & & & \\ \cdot & & & & & & & & \cdot & & \\ \cdot & & & & & & & & & \cdot & 0 \\ 0 & & & & & & & & & & \beta_k \end{bmatrix}, \quad Q(t+1)=Q(t). \tag{2.42}$$

Consequently, a typical column vector $\underline{\psi}_k(t)$ corresponding to an eigenvalue β_1 of multiplicity $\underline{m}$ is expressed as

$$\underline{\psi}_k(t) = \left[\frac{t^{k-1}}{(k-1)!}\underline{q}_1(t) + \frac{t^{k-2}}{(k-2)!}\underline{q}_2(t) + \ldots + t\underline{q}_{k-1}(t) + \underline{q}_k(t)\right] e^{\beta_1 t}, \tag{2.43}$$

$k = 1, 2, \ldots m$.

For the stationary linear system with a repeated eigenvalue of multiplicity m, the corresponding column vector has the form

$$\underline{\psi}_k(t) = \left[\frac{t^{k-1}}{(k-1)!}\underline{c}_1 + \frac{t^{k-2}}{(k-2)!}\underline{c}_2 + \ldots + t\underline{c}_{k-1} + \underline{c}_k\right] e^{\lambda_k t},$$

where $\underline{c}_i$ is the constant vector with the components: $c_{ij} = 1$ for $j \leq i$ and $c_{ij} = 0$ for $j > i$.

6. Stability criteria for the linear system with periodic coefficients shown below may be derived most readily from the form of the fundamental matrix.

$$\dot{\underline{x}} = A(t)\underline{x} + \underline{b}(t), \quad \underline{x}(t_o) \neq 0, \quad t_o \leq t < \infty, \tag{2.44}$$

$$A(t+1) = A(t) \quad \text{and} \quad ||\underline{b}|| \leq N,$$

where N is a finite, positive, real number.

(a) The solution of the above equation, $\underline{x}(t)$, is bounded for all $t \geq t_o$ if, and only if, all the characteristic exponents β_i have negative real parts [Ref. 33, pp. 23-24].

Proof: The solution $\underline{x}(t)$ is written as shown previously from Eq. (2.16):

$$\underline{x}(t) = \Phi(t)\Phi^{-1}(t_o)\underline{x}(t_o) + \int_{t_o}^{t} \Phi(t)\Phi^{-1}(\tau)\underline{b}(\tau)d\tau \ .$$

Substituting Eq. (2.34) for $\Phi(t)$, we obtain the following form of solution for $\underline{x}(t)$:

$$\underline{x}(t) = \Pi(t)e^{B(t - t_o)}\Pi^{-1}(t_o)\underline{x}(t_o) + \int_{t_o}^{t} \Pi(t)e^{B(t - \tau)}\Pi^{-1}(\tau)\underline{b}(\tau)d\tau \ . \tag{2.45}$$

It is clear that the first term will decay exponentially with time, if all the characteristic exponents (eigenvalues of B) have negative real parts. The second term may be written as

$$\int_{t_o}^{t} \Pi(t)e^{B(t - \tau)}\Pi^{-1}(\tau)\underline{b}(\tau)d\tau = \Pi(t)\int_{t_o}^{t} e^{B(t - \tau)}\underline{u}(\tau)d\tau$$
$$= \Pi(t)\underline{y}(t) \ .$$

$\underline{y}(t)$ is a familiar convolution integral. Its components are bounded if the eigenvalues of B have negative real parts. Note that $\Pi^{-1}(t)$ is bounded by the fact that $\Pi(t)$ is non-singular Q.E.D.

(b) Since the characteristic exponents and the characteristic roots are related by Eq. (2.36), an alternative stability criterion may be stated in terms of the constraint on the characteristic roots:
The solution of Eq. (2.44), $\underline{x}(t)$, is bounded for all $t \geq t_o$, if, and only if, all the characteristic roots z_i lie within the unit circle. The proof follows directly from Eq. (2.36) and the preceding stability criterion stated above.

7. It is not easy, in general, to calculate analytically the fundamental matrix $\Phi(t)$, even if we know its form is given by Eq. (2.34). Since stability is often the most important consideration in many physical problems, the calculation of characteristic exponents is a matter of great importance. In order to calculate the characteristic exponents, however, it is necessary to know the fundamental matrix $\Phi(t)$ <u>only at $t = 1$</u> as previously shown by Eqs. (2.23) and (2.36). Since $\Phi(t)$ must satisfy the matrix equation,

$$\dot{\Phi}(t) = A(t)\Phi(t), \quad \Phi(0) = I, \quad 0 \leq t \leq 1 ,$$
$$A(t + 1) = A(t) , \tag{2.46}$$

we need to integrate the above equation only over the fundamental period, $0 \leq t \leq 1$, to obtain $\Phi(1) = C$. Because we only need the numerical solution $\Phi(1)$, rather than the analytical solution $\Phi(t)$, this can be obtained to any desired accuracy by the use of a digital computer. Having obtained the numerical matrix $\Phi(1)$, we simply compute the eigenvalues of $\Phi(1)$ to determine the characteristic roots. The eigenvalue analysis is one of the most common computer routines. Perhaps the simplest and most straightforward method for the numerical integration of Eq. (2.46) by a digital computer would be the method of mean coefficients [Ref. 34]. This method is essentially an approximation of the periodic coefficient matrix $A(t)$ by a finite number of constant coefficient matrices within the fundamental period, $0 \leq t \leq 1$. The period is divided into m subintervals:

$$\sum_{k=1}^{m} T_k = 1 ,$$

$$T_k = t_k - t_{k-1}, \quad k = 1, 2, \dots m ,$$

$$t_o = 0 , \quad t_m = 1 .$$

If within each subinterval, T_k, the elements $a_{ij}(t)$ of the matrix $A(t)$ are replaced by their respective mean values, we have

$$A(t) = A_k , \quad 0 \leq t \leq T_k , \quad k = 1, 2, \dots m , \tag{2.47}$$

$$a_{ij}(T_k) = \frac{1}{T_k}\int_o^{T_k} a_{ij}(t)dt .$$

Now we solve the following equations in place of Eq. (2.46):

$$\dot{\Phi}(t) = A_k\Phi(t) , \quad \Phi(0) = I , \quad 0 \leq t \leq T_k , \tag{2.48}$$
$$k = 1, 2, \dots m .$$

The solution at $t = 1$ is given by "cascading" the solutions of the above equations:

$$\Phi(1) = \prod_{k=1}^{m} \exp A_k T_k \tag{2.48a}$$

This entire numerical calculation is quite practical for a modern computer; the answer is obtained in a matter of minutes.

8. We summarize briefly the main results of the classical Floquet theory:

   (a) The solution of the linear differential equation with continuous, periodic coefficients, Eq. (2.20), is completely determined for all time $t \geq 0$ if the solution is known only over the fundamental period $0 \leq t \leq 1$.

   (b) Neither the solution nor the fundamental matrix $\Phi(t)$ can be analytically calculated in general, even over the fundamental period.

   (c) The characteristic exponents and characteristic roots are determined completely by the numerical matrix $\Phi(1)$. This can be computed to any desired accuracy by a digital computer in a matter of minutes.

   (d) Consequently, the stability of a linear system with periodic coefficients and its natural response at the discrete instants of time separated by a full period, may be determined within a desired accuracy by machine computation, using Eq. (2.25a).

We have listed some of the most important properties of linear systems with continuous, periodic coefficients and discussed to a certain extent the physical significance and computational aspects of the characteristic exponents. Next we shall discuss an extension of the Floquet theory to the linear systems with piece-wise continuous periodic coefficients.

3. Extension of the Classical Theory

There are many examples of modern engineering systems in which one or more system parameters may vary periodically but not always continuously. For instance, periodically operated on-off type switches are used in many communication and control systems and a large number of such systems may be represented by linear differential equations with piece-wise continuous periodic coefficients. This class of systems will be called piece-wise, continuous, periodic systems. Very fortunately, it is possible to extend the conclusions of the classical Floquet theory to the above class of systems without elaborate modifica-

tions. We shall first treat a special case of a linear system with piece-wise constant, periodic coefficients and proceed afterwards to a discussion of linear systems with piece-wise continuous, periodic coefficients.

a. We consider a generalized Hill-Meissner equation to illustrate the theory for piece-wise continuous periodic systems.

$$\begin{aligned}
&\dot{\underline{x}} = A(t)\underline{x}\ , \quad 0^{+} \leq t < \infty, \\
&A(t+1) = A(t): \ n \times n \ \text{matrix}, \quad t \neq k, \quad t \neq k + 1/2, \\
&A(t) = A_1, \quad k < t < k + 1/2, \\
&A(t) = A_2, \quad k + 1/2 < t < k + 1, \\
&k = 0, 1, 2, \ldots
\end{aligned} \tag{2.49}$$

The system is switching between the two constant parameter systems with coefficient matrices A_1 and A_2 respectively. In the one-dimensional case, this may be visualized as shown in Fig. 7. The solution, $\underline{x}(t)$, of the piece-wise constant periodic system, Eq. (2.49), is not necessarily discontinuous. This can be seen from the one-dimensional example shown in Fig. 4. If we assume the modulator $m(t)$ is a square wave of the form shown in Fig. 7, then we have $\dot{x} + a(t)x = 0$, $0 \leq t < \infty$,

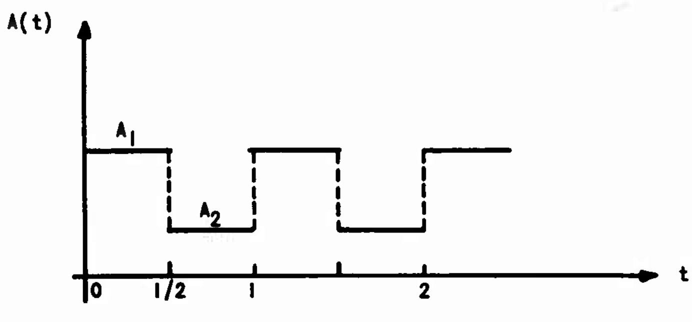


FIG. 7. A PIECE-WISE CONSTANT PERIODIC COEFFICIENT (FOR A ONE-DIMENSIONAL SYSTEM).

$t \neq k$, $t \neq k + 1/2$, where $a(t)$ is piece-wise continuous. The solution is $x(t) = x(0)\exp[-\int_0^t a(\tau)d\tau]$, and it is continuous for $t \geq 0$. In general, we make the following observations on the continuity of the solution of Eq. (2.49):

1. It is always possible, at least in principle, in electrical and mechanical systems to choose the stored charges and flux linkages and the canonical coordinates and momenta as the state variables, respectively. Then these variables are continuous, and are differentiable almost everywhere because of the basic conservation laws such as conservation of charge, flux linkage and momenta etc., provided the delta function and its derivatives are excluded.

2. If it is not convenient to choose the state variables as specified above for some physical reasons, such as difficulty of measurements and observability, then the solution $\underline{x}(t)$ of Eq. (2.49) may not be continuous at the points of discontinuity. In such a case, we have

$$\underline{x}(k + 1/2^+) = S_1\underline{x}(k + 1/2^-) \,, \qquad (2.50)$$
$$\underline{x}(k^+) = S_2\underline{x}(k^-)$$
$$k = 0, 1, 2, \ldots$$

where S_1 and S_2 are nonsingular constant matrices. The above equations will be referred to as the boundary conditions. Normally these boundary conditions are given or derived from the constraints such as conservation of momentum, charge and flux linkage, etc.

3. Equation (2.49) plus the boundary condition, Eq. (2.50), are needed for representation of the type of piece-wise continuous, linear periodic system under consideration. Now we write down the solution.

$$\underline{x}(t) = \Phi(t, k^+)\underline{x}(k^+), \quad t \neq k, \quad t \neq k + 1/2 \,, \qquad (2.51)$$
$$k = 0, 1, 2, \ldots$$

Because of the periodicity of $A(t)$, we have the matrix difference equation

$$\Phi(t + 1, k^+) = \Phi(t, k^+)C \qquad (2.52)$$

and the vector difference equation

$$\underline{x}(k + 1^+) = C\underline{x}(k^+) \,. \qquad (2.53)$$

The latter can be derived easily from Eqs. (2.51) and (2.52). If we set $k = 0$, Eq. (2.53) becomes $\underline{x}(1^+) = C\underline{x}(0^+)$ and it is necessary to solve Eq. (2.49) plus Eq. (2.50) only for $k = 0$ to obtain the discrete transition matrix. We solve a differential equation with constant coefficients successively in each subinterval, $0 < t < 1/2$ and $1/2 < t < 1$, to obtain

$$C = S_2 \exp A_2(1/2) S_1 \exp A_1(1/2) \tag{2.53a}$$

It is interesting to note that all the characteristic roots of C may still be within the unit circle even if some of the eigenvalues of A_1 and A_2 have positive real parts. Conversely, some of the characteristic roots of C may be outside the unit circle even if all the eigenvalues of A_1 and A_2 have negative real parts. The latter is illustrated by the following example:

Example 2:

For the sake of simplicity, we set $S_1 = S_2 = I$ and choose $\exp A_2(1/2)$ and $\exp A_1(1/2)$ to be

$$\exp A_1(1/2) = \begin{bmatrix} .4 & 10 \\ 0 & .2 \end{bmatrix}, \qquad \exp A_2(1/2) = \begin{bmatrix} .4 & 0 \\ 10 & .2 \end{bmatrix}.$$

A_1 and A_2 have the same eigenvalues, both negative real: $p_1 = -2 \ln 2.5$, $p_2 = -2 \ln 5$

$$C = \begin{bmatrix} 100.16 & 2 \\ 2 & .04 \end{bmatrix} \quad \text{and Trace } C = 100.2 = z_1 + z_2 \,.$$

Clearly either z_1 or z_2 must be outside the unit circle.

4. If the time origin were not $t_o = 0$, but $t_o = 1/4$ in Eq. (2.49), and all other things were to remain the same, then we could also show easily,

$$\underline{x}(t) = \Phi(t, 1/4)\underline{x}(1/4) \,, \tag{2.54}$$

$$\Phi(t + 1, 1/4) = \Phi(t, 1/4)C' \,, \tag{2.55}$$

$$\underline{x}(r + 1) = C'\underline{x}(r) \,, \tag{2.56}$$

$$r = 1/4,\ 1 + 1/4 \,, \ldots$$

$$= k + 1/4, \quad k = 0, 1, 2, \ldots$$

$$\text{and } C' = \exp A_1(1/4) C \exp A_1(-1/4) \,. \tag{2.57}$$

The two discrete transition matrices C and C' are related by a similarity transformation as in the previous case of classical theory.

The above results can be extended directly to the general case in which there are a finite number of discontinuities of $A(t)$ in the fundamental period $0 \leq t \leq 1$. Next we discuss the general piece-wise continuous periodic system which includes the above special case.

b. A linear system with piece-wise continuous, periodic coefficients may be defined by the following equation:

$$\begin{aligned} &\dot{\underline{x}} = A(t)\underline{x}, \quad 0^+ \leq t < \infty, \quad t \neq t_i + k \;, \\ &A(t + 1) = A(t), \\ &A(t) = A_i(t), \quad k + t^+_{i-1} \leq t \leq k + t^+_i \;, \\ &i = 1, 2, \ldots m; \quad k = 0, 1, 2, \ldots \end{aligned} \tag{2.58}$$

where $T_i = t^-_i - t^+_{i-1}$, $\sum_{i=1}^{m} T_i = 1$, $t_o = 0$ and $t_m = 1$, and by the boundary conditions

$$\begin{aligned} &\underline{x}(t^+_i + k) = S_i\underline{x}(t^-_i + k) \;, \\ &i = 1, 2, \ldots m; \quad k = 0, 1, 2, \ldots \end{aligned} \tag{2.59}$$

where S_i are constant, nonsingular $n \times n$ matrices.
The one-dimensional case with three discontinuities may be sketched as shown in Fig. 8.

1. The solution may be written as

$$\begin{aligned} &\underline{x}(t) = \Phi(t, k^+)\underline{x}(k^+), \quad t \neq k + t_i, \quad t \geq k^+, \\ &i = 1, 2, \ldots m. \\ &k = 0, 1, 2, \ldots \end{aligned} \tag{2.60}$$

Because of the periodicity of $A(t)$, we have the matrix difference equation

$$\Phi(t + 1, k^+) = \Phi(t, k^+)C \tag{2.61}$$

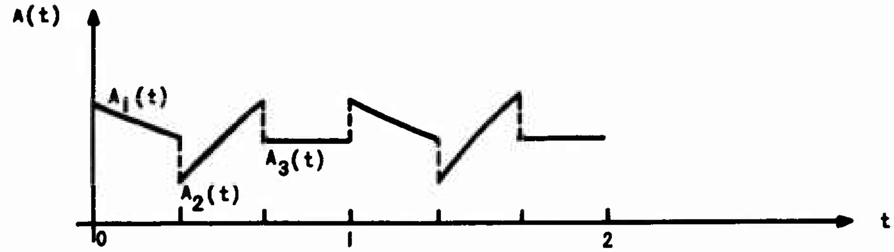


FIG. 8. A PIECE-WISE CONTINUOUS PERIODIC COEFFICIENT (FOR A ONE-DIMENSIONAL SYSTEM).

and the vector difference equation

$$\underline{x}(k + 1^+) = C\underline{x}(k^+) \ . \tag{2.62}$$

It is seen from equation (2.61) that

$$C = \Phi(1^+, 0^+) \equiv \Phi(1^+) \ , \tag{2.62a}$$

and we need to solve the matrix equations associated with Eqs. (2.58) and (2.59) in order to calculate the matrix $\Phi(1^+)$.

$$\dot{\Phi}(t) = A(t)\Phi(t), \quad \Phi(0^+) = I, \quad 0^+ \leq t \leq 1^+, \tag{2.63}$$

$$A(t) = A_i(t), \quad t_{i-1} < t < t_i,$$

$$i = 1, 2, \ldots m; \quad k = 0, 1, 2, \ldots$$

where $t_o = 0$, and $t_m = 1$.
The boundary conditions are derived from Eq. (2.59),

$$\Phi(t_i^+) = S_i\Phi(t_i^-) \ , \tag{2.64}$$

$$i = 1, 2, \ldots m \ .$$

The solution at $t = 1^+$, $\Phi(1^+)$ has the form

$$\Phi(1^+) = S_m\Phi_{m-1}(t_m, t_{m-1})S_{m-1} \ldots \Phi_o(t_1, 0^+) = C \ , \tag{2.65}$$

and $\Phi_i(t, t_i)$ is the fundamental matrix associated with the ith equation of (2.62).
The determinant of C is obtained from Eqs. (2.26) and (2.65).

$$\det C = [\prod_{i=0}^{m} \det S_i] \exp \int_0^1 \operatorname{Tr} A(\tau)d\tau$$

$$= z_1 z_2 \ldots z_n \tag{2.66}$$

where z_i are the eigenvalues of C (characteristic roots).

2. The analytical form of the fundamental matrix may be found from the matrix difference equation (2.61) and the boundary condition, Eq. (2.64). If we assume that the fundamental matrix has the same form as shown by Eq. (2.34) in the classical Floquet theory, then we obtain

$$\Phi(t) = P(t)e^{Bt}, \quad t > 0 , \tag{2.67}$$

$$P(t + 1) = P(t) ,$$

$$\text{and} \quad P(t_i^+) = S_i P(t_i^-) , \tag{2.68}$$

$$i = 1, 2, \ldots m .$$

The discontinuities in the fundamental matrix $\Phi(t)$ are transferred to the periodic matrix $P(t)$ since e^{Bt} is a continuous matrix. For the linear system with piece-wise constant periodic coefficients with m discontinuities, Eq. (2.65) becomes

$$\Phi(1^+) = S_m \exp A_{m-1}T_{m-1}S_{m-1} \ldots \exp A_o T_o ,$$

where $\Phi(t) = \exp A_i t$; $T_i = t_i - t_{i-1}$.
This was to be expected.
It is clear from Eqs. (2.65) and (2.67) that the characteristic roots and the characteristic exponents are exactly the same as in the classical case, and they are related by the same equations as Eqs. (2.35) to (2.38). The boundary conditions, Eq. (2.68), are the only difference caused by the discontinuities.

3. One can derive the stability criterion for the linear system with piece-wise continuous periodic coefficients in the same manner as for the classical case. We consider Eq. (2.58) with a finite forcing term,

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + \underline{b}(t) , \quad 0 \leq t < \infty , \tag{2.69}$$

$$\text{where} \quad ||b(t)|| \leq b_o \quad \text{for} \quad t \geq 0 ,$$

with the same boundary conditions as given by Eq. (2.59). Now we state the following theorem as an extension of the classical Floquet theory:
The solution $\underline{x}(t)$ of the linear system with piece-wise continuous periodic coefficients, Eq. (2.69), is stable if, and only if, all the characteristic roots of the force-free system, Eq. (2.58) plus Eq. (2.59), are within the unit circle. (See Appendix A for proof of this theorem.)

4. In summary, the linear system with piece-wise continuous periodic coefficients is completely described by the differential equation and the boundary conditions at the points of discontinuity. The periodicity of the transmission characteristics as shown by Eqs. (2.61) and (2.67), and the stability criterion stated above, are direct extensions of the results of the classical Floquet theory presented earlier. In short, we can handle the linear system with piece-wise continuous periodic coefficients in almost the same way as we handle the linear system with continuous periodic coefficients.

4. Calculations of Forced Responses and Transfer Functions

Even if we know from the Floquet theory that the fundamental matrix of the linear system with periodic coefficients has the form $\Phi(t) = \Pi(t)\exp Bt$, it is very difficult in general to find the periodic matrix $\Pi(t)$. Consequently, the form of solution given by Eq. (2.45) is not suitable to practical applications. If we want to obtain a practical solution of the inhomogeneous vector equation (2.44), we have to use something other than Eq. (2.45).

a. Since many time functions such as exponentials, ramps and sinusoids can be generated as solutions of linear differential equations with constant coefficients, one can "simulate" a large class of forcing functions by appropriate stationary linear systems excited by proper initial conditions [Ref. 7]. Assuming that the input $\underline{u}(t)$ belongs to this class, we obtain in general the following equations:

$$\dot{\underline{u}} = W\underline{u}, \qquad 0 \le t_o \le t < \infty , \tag{2.70}$$

$$\dot{\underline{x}} = A(t)\underline{x} + F\underline{u}, \qquad 0 \le t_o \le t < \infty , \tag{2.71}$$

$$A(t + 1) = A(t): \quad n \times n \text{ matrix} ,$$

where W is a constant $m \times m$ matrix, $\underline{u}(t)$ is an $m \times 1$ input vector and F is a constant $n \times m$ matrix.

The forcing term in Eq. (2.71) may be set as $F\underline{u}(t) = \underline{b}(t)$ to make Eq. (2.71) identical with Eq. (2.69); but we shall use the above forms which are more widely seen in the literature.

Equations (2.70) and (2.71) may be combined into a single equation

$$\begin{bmatrix} \dot{\underline{u}} \\ \dot{\underline{x}} \end{bmatrix} = \begin{bmatrix} W & 0 \\ F & A(t) \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{x} \end{bmatrix}, \quad 0 \le t_o \le t < \infty ; \tag{2.72}$$

or we may rewrite the expression for simplicity as

$$\dot{\underline{v}} = H(t)\underline{v} , \quad t_o \le t < \infty , \tag{2.73}$$

where $H(t + 1) = H(t) : (m + n) \times (m + n)$ matrix.

Thus we have transformed the inhomogeneous n-dimensional vector equation (2.71) into the homogeneous $(m + n)$-dimensional vector equation (2.72). Since the latter is a linear differential equation with periodic coefficients, its solution is

$$\underline{v}(t) = \Phi(t, t_o)\underline{v}(t_o) = \Phi(t)\Phi^{-1}(t_o)\underline{v}(t_o) , \tag{2.74}$$

where $\Phi(t + 1) = \Phi(t)C$.

The vector difference equation is derived as shown previously:

$$\underline{v}(i + 1) = \Phi(t_o)C\Phi^{-1}(t_o)\underline{v}(i) = R\underline{v}(i) , \tag{2.75}$$

$$i = k + t_o, \quad k = 0, 1, 2, \ldots$$

$$\text{or} \quad \underline{v}(k + 1 + t_o) = R\underline{v}(k + t_o) ,$$

and the solution is

$$\underline{v}(k + t_o) = R^k\underline{v}(t_o) . \tag{2.76}$$

It is a straightforward matter, therefore, to calculate the "sampled" response of a linear system with periodic coefficients, provided that the external input can be generated as the solution of an appropriate

linear differential equation with constant coefficients.

b. Even though the inhomogeneous equation with periodic coefficients, Eq. (2.69), may be transformed into the homogeneous difference equation with constant coefficients, Eq. (2.75), it is not possible in general to describe this system completely by a transfer function independent of time. This can be seen from the fact that the discrete transition matrix R depends on the time of excitation t_o in Eq. (2.75).

1. We start from Eq. (2.70) in order to derive the transfer function. It is clear from Eq. (2.70) that the fundamental matrix $\Phi(t, t_o)$ has the form

$$\Phi(t, t_o) = \begin{bmatrix} e^{W(t-t_o)} & 0 \\ \Phi_{21}(t, t_o) & \Phi_{22}(t, t_o) \end{bmatrix}, \tag{2.77}$$

where $\Phi_{22}(t, t_o)$ is the fundamental matrix associated with the equation $\dot{\underline{x}} = A(t)\underline{x}$, and $\Phi_{21}(t, t_o)$ is the "coupling term" due to the external input and has the form

$$\Phi_{21}(t, t_o) = \int_{t_o}^{t} \Phi(t)\Phi^{-1}(\tau)e^{W(t-t_o)}d\tau \ .$$

The form of the discrete transition matrix R can be inferred readily from Eqs. (2.74), (2.75) and (2.77).

$$R = \begin{bmatrix} R_{11}(t_o) & 0 \\ R_{21}(t_o) & R_{22}(t_o) \end{bmatrix} = R(t_o) \ . \tag{2.78}$$

In order to define the transfer function, we set $t_o = 0$. Then Eq. (2.75) becomes

$$\underline{v}(k + 1) = R\underline{v}(k) \ ,$$

and taking the z-transform of this equation we obtain

$$\underline{v}(z) = [zI - R]^{-1}\underline{v}(0^+) \ , \tag{2.79}$$

or

$$\begin{bmatrix} \underline{u}(z) \\ \underline{x}(z) \end{bmatrix} = z \begin{bmatrix} (zI - R_{11})^{-1} & 0 \\ (zI - R_{22})^{-1}R_{21}(zI - R_{11})^{-1} & (zI - R_{22})^{-1} \end{bmatrix} \begin{bmatrix} \underline{u}(0^+) \\ \underline{x}(0^+) \end{bmatrix}$$

It follows from this equation that

$$\underline{x}(z) = z(I - R_{22})^{-1}R_{21}\underline{u}(z) + z(zI - R_{22})^{-1}\underline{x}(0^+) \ . \qquad (2.80)$$

The transfer function $T_{ij}(z)$ is defined as

$$T_{ij}(z) = \frac{x_j(z)}{u_i(z)} \ , \qquad (2.81)$$

with $\underline{x}(0^+) = \underline{0}$ in Eq. (2.80).

It can be seen from Eq. (2.80) that R_{22} is the discrete transition matrix of the homogeneous equation $\dot{\underline{x}} = A(t)\underline{x}$, if $\underline{u}(t) = \underline{0}$ in Eq. (2.71). Consequently, the poles of the transfer function which are the characteristic roots of R_{22} are independent of the initial time, t_o. But the zeros will depend on t_o because both R_{21} and R_{22} are functions of t_o. Therefore one needs a time-dependent transfer function $T_{ij}(z, t_o)$, to describe the system, Eq. (2.72), completely. It is sufficient, however, to limit the range of t_o within the fundamental period $0 \le t_o \le 1$. This can be seen from the periodicity of state transition characteristics as shown by Eqs. (2.74) and (2.75). Consequently, the time-dependent transfer function $T_{ij}(z, t_o)$ is analogous to the modified Z-transform [Ref. 35] characterization of sampled-data systems.

2. One can give a simpler, alternate derivation of the transfer function by starting from the integral representation of the forced response, Eq. (2.45). If we set $\underline{x}(t_o) = \underline{0}$, then the solution of the inhomogeneous equation (2.71) is

$$\underline{x}(t) = \int_{t_o}^{t} \Phi(t)\Phi^{-1}(\tau)F\underline{u}(\tau)d\tau \ . \qquad (2.82)$$

Since $\underline{u}(t)$ is a solution of the differential equation (2.70), we have $\underline{u}(t) = e^{W(t-t_o)}\underline{u}(t_o)$. Substituting this expression into the above equation, we obtain

$$\underline{x}(t) = \Phi(t)\int_{t_o}^{t} \Phi^{-1}(\tau)Fe^{W(\tau-t_o)}\underline{u}(t_o)d\tau , \qquad (2.83)$$

and $$\underline{x}(i + 1) = C\underline{x}(i) + C[\int_o^1 \Phi^{-1}(\tau)Fe^{W\tau}d\tau]e^{W(k-t_o)}\underline{u}(t_o)$$
$$= C\underline{x}(i) + D\underline{u}(i) , \qquad (2.84)$$

$$i = t_o + k, \quad k = 0, 1, 2, \ldots$$

where $D = C \int_o^1 \Phi^{-1}(\tau)Fe^{W\tau}d\tau.$ (2.85)

Applying the Z-transform to the above equation, we get

$$\underline{x}(z) = [zI - C]^{-1} D\underline{u}(z) , \qquad (2.86)$$

and the transfer function can be defined in the same way as Eq. (2.81). The theoretical simplicity of this method is offset, however, by the practical difficulty in evaluating the integral, Eq. (2.85).

III. APPLICATIONS TO SYSTEM ANALYSIS AND DESIGN

A. INTRODUCTION

We have developed in the previous chapter the basic mathematical tools needed for analysis of linear systems with periodic coefficients (or parameters). So far we have discussed the representation of this class of systems by linear differential equations. In this discussion, the main emphasis has been placed on the extension of Floquet theory to cover not only linear systems with continuous periodic parameters, but also linear systems with piece-wise continuous periodic parameters. Such systems include many modern engineering systems.

Now we are in a position to use these tools in solving practical engineering problems. In this chapter we apply the extended Floquet theory to the exact stability analysis of feedback amplifiers with periodically time-varying parameters. The nature of design problems and limited design method based on the Floquet theory and frequency domain approximations are discussed by means of an example. We review briefly the basic features of a simple carrier frequency feedback system in order to be better prepared for detailed discussions of more realistic examples later.

The system model shown in Fig. 9 may represent several types of physical systems, but in this case we will take it to be a carrier D.C.

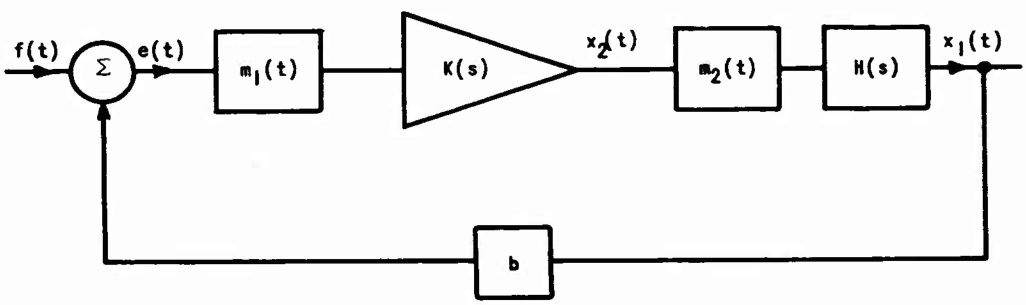


FIG. 9. THE BASIC MODEL OF CARRIER FREQUENCY FEEDBACK SYSTEM.

amplifier. The forward transmission link represents a standard form of chopper modulated D.C. amplifier. The low frequency (0 to about 100 cps) signal $e(t)$ is modulated by a high frequency carrier. This is done by a product-type modulator, $m_1(t)$. The modulated signal $e(t)m_1(t)$ is amplified by an A.C. amplifier $K(s)$ and demodulated by another product-type modulator $m_2(t)$. Normally $m_1(t) = m_2(t)$, but this need not always be the case. The demodulated signal $x_2(t)m_2(t)$ is passed through a low-pass filter $H(s)$ to attenuate the unwanted high frequency modulation products. This chopper-stabilized amplifier is almost free of zero drift, which is the major problem in ordinary D.C. amplifiers. The feedback is used to maintain a constant transmission from input to output despite variations in operating characteristics of the components due to aging or environmental changes. The basic model shown in Fig. 9 includes the most essential features only; there are many variations of this model in practice.

B. ANALYSIS OF FEEDBACK SYSTEMS WITH PIECE-WISE CONTINUOUS PERIODIC PARAMETERS

1. The System Model and Basic Equations

Now we take up a more realistic carrier D.C. amplifier and demonstrate the power of the extended Floquet theory in an analysis of the 4th order periodic system shown in Fig. 10. The system configuration is very similar to the one shown in Fig. 9 except that the system in Fig. 10 has a more complicated feedback link. The A.C. amplifier in the forward link is described by a second-order transfer function. The modulator-demodulator pair is introduced in the feedback link to provide a greater degree of isolation between the input and the output. The two modulators $m_1(t)$ and $m_2(t)$ are both periodic functions having a period 1, as sketched in Fig. 11. The precise waveforms of $m_1(t)$ and $m_2(t)$ are plotted from experimental data in Fig. 11b. Neither $m_1(t)$, a piece-wise continuous periodic function, nor $m_2(t)$, a continuous periodic function, is differentiable at $t = k$ and $t = k + 1/2$, $k = 0, 1, 2, \ldots$.

Due to the presence of the two modulators, the system of Fig. 10

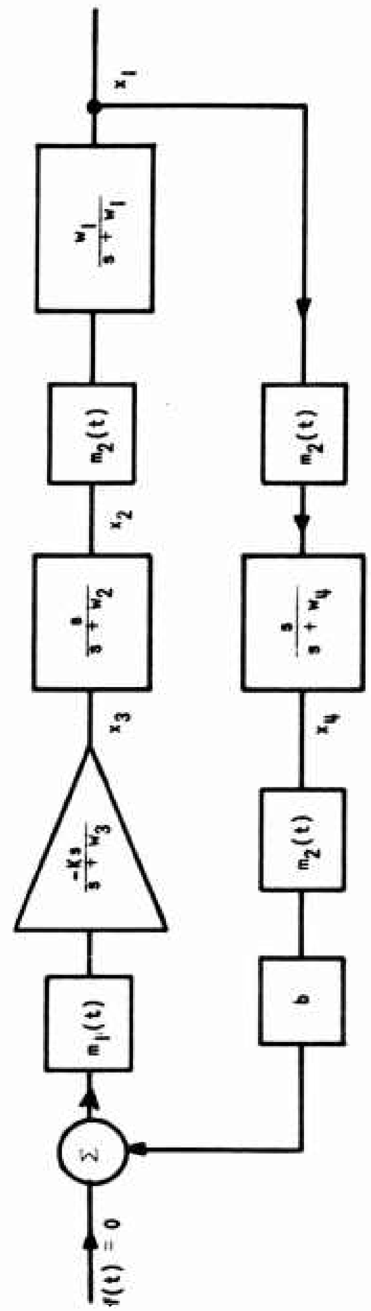
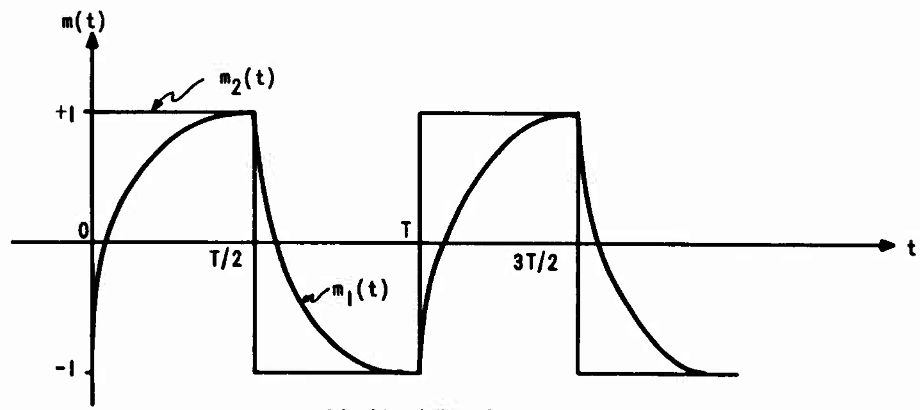
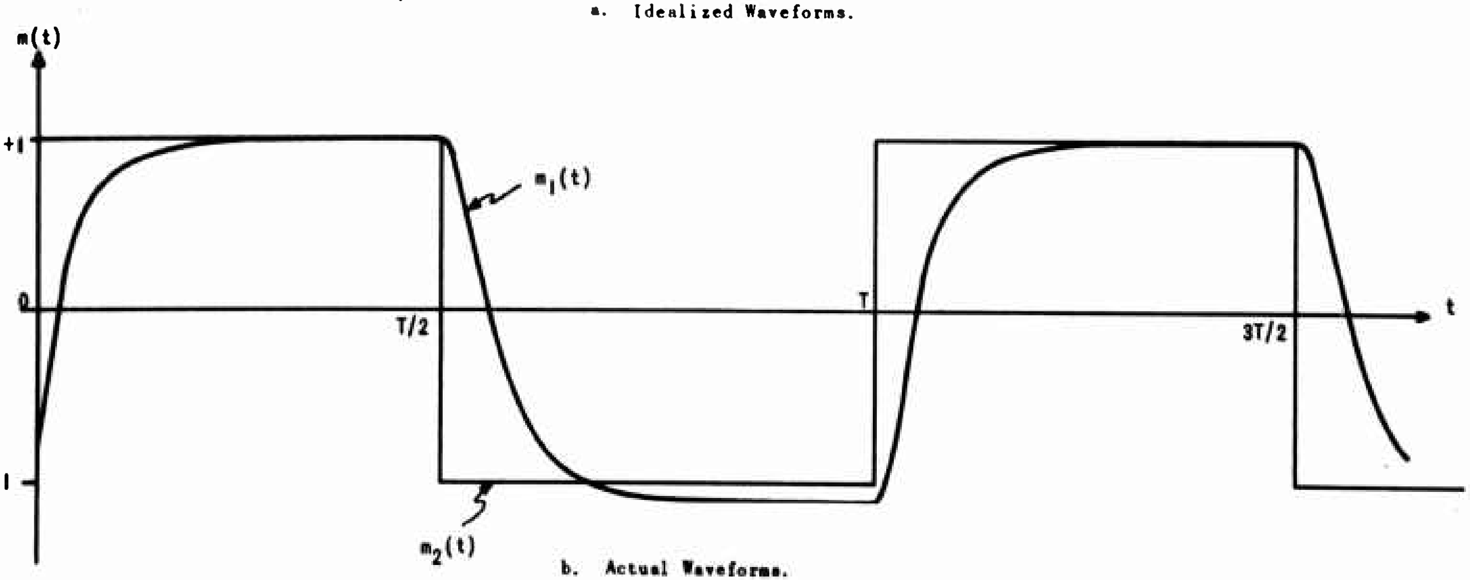


FIG. 10. THE SYSTEM MODEL.



a. Idealized Waveforms.



b. Actual Waveforms.

FIG. 11. MODULATION WAVEFORMS FOR THE SYSTEM OF FIG. 10.

is represented by a linear differential equation with piece-wise continuous periodic coefficients. Although this may be a special example, the method of analysis is general enough, so that it should be clear from this example how to proceed with other types of piece-wise continuous, linear periodic systems. The system equations can be obtained directly from the model shown in Fig. 10 as follows:

$$\left.\begin{aligned}(p + w_1)x_1 &= w_1 m_2 x_2 ,\\ (p + w_2)x_2 &= px_3 ,\\ (p + w_3)x_3 &= p(-Kbm_1 m_2 x_4) ,\\ (p + w_4)x_4 &= p(m_2 x_1) ,\end{aligned}\right\} \tag{3.1}$$

where $m_1(t + 1) = m_1(t)$; $m_2(t + 1) = m_2(t)$; $m_2^2 = 1$; $p = \frac{d}{dt}$; $t > 0$; and some $x_i(0^+)$ are non-zero.

Since $m_1(t)$ and $m_2(t)$ are not always differentiable, and since the solutions of Eq. (3.1) are not necessarily continuous at the discontinuities of $m_1(t)$ and $m_2(t)$, we rewrite Eq. (3.1) in a more convenient form by introducing a change of the state variables, as shown by the nonsingular transformation given below:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & +Kbm_1 m_2 \\ -m_2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{3.2}$$

or

$$\underline{v}(t) = P(t)\underline{x}(t)$$

and

$$P(t + 1) = P(t) .$$

The linear transformation, Eq. (3.2), provides a two-fold advantage. First, we avoid the differentiation of $m_1(t)$ and $m_2(t)$; and second, the new set of state variables v_i physically correspond to the voltages

across the capacitors if we draw the electrical analogue, Fig. 12, of the system shown in Fig. 10.

It follows from the basic laws governing electromagnetic phenomena that the voltages across the capacitors must be continuous provided that no impulse is allowed.

Combining Eqs. (3.1) and (3.2) we obtain the system equation in a standard form:

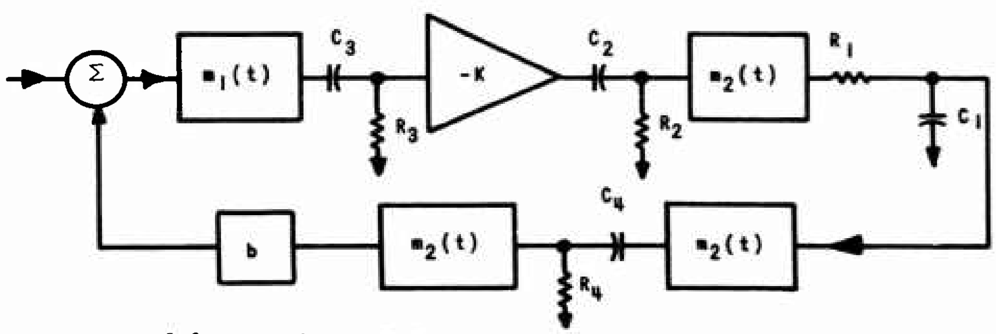
$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{bmatrix} = \begin{bmatrix} -w_1(Kbm_1m_2+1) & w_1m_2 & w_1m_2 & -w_1Kbm_1 \\ +w_2Kbm_1 & -w_2 & -w_2 & +w_2Kbm_1m_2 \\ +w_3Kbm_1 & 0 & -w_3 & +w_3Kbm_1m_2 \\ -w_4m_2 & 0 & 0 & -w_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \tag{3.3}$$

and

$$\left.\begin{aligned} &\underline{\dot{v}}(t) = A(t)\underline{v}(t) , \quad t > 0 , \quad t = k + t_i , \\ &A(t + 1) = A(t) , \\ &A(t) = A_i(t) , \quad k + t_i^+ \leq t \leq k + t_{i+1}^- , \\ &i = 0, 1, 2 ; \quad k = 0, 1, 2, \ldots \end{aligned}\right\} \tag{3.4}$$

where $t_o = 0$, $t_1 = 1/2$, and $t_2 = 1$.

This is the same type of equation as Eq. (2.61) discussed previously in the extended Floquet theory. The boundary conditions are



$w_i = R_iC_i$, $i = 1, 2, 3, 4$

FIG. 12. AN ELECTRICAL ANALOGUE OF THE CARRIER SYSTEM.

obtained from the fact that the solution $\underline{v}(t)$ must be continuous because of its physical interpretation mentioned above.

$$\underline{v}(k^+) = \underline{v}(k^-) = \underline{v}(k) \ , \tag{3.5}$$
$$\underline{v}(k + 1/2^-) = \underline{v}(k + 1/2^+) = \underline{v}(k + 1/2) \ .$$

The solution is of the form

$$\underline{v}(t) = \Phi(t)\underline{v}(0^+) \ , \quad \Phi(0^+) = I \ , \quad t > 0 \ . \tag{3.6}$$

Because of the periodicity of $A(t)$, we have

$$\Phi(t + 1) = \Phi(t)C \ , \quad t > 0 \ , \tag{3.7}$$

where $C = \Phi(1^+)$

and

$$\underline{x}(k + 1^+) = C\underline{x}(k^+) \ , \quad k = 0, 1, 2, \ldots \tag{3.8}$$

Since the coefficient matrix $A(t)$ is not piece-wise constant but piece-wise continuous, it is not possible to calculate analytically the fundamental matrix $\Phi(t)$. One can compute, however, the discrete transition matrix C to any desired accuracy by a digital computer.

2. Calculation of the Characteristic Roots and Stability Analysis

It was shown by the extended Floquet theory in the previous chapter that stability of the linear system with piece-wise continuous periodic coefficients, Eq. (3.3), is completely determined by the eigenvalues of the discrete transition matrix C. These eigenvalues are called the characteristic roots of the system.

a. Calculation of Characteristic Roots

The constant matrix C is obtained from numerical integration of the matrix equation associated with Eq. (3.3) over the fundamental period:

$$\dot{\Phi}(t) = A(t)\Phi(t) \ , \quad \Phi(0^+) = I \ , \quad 0^+ \leq t \leq 1^+ \ , \tag{3.9}$$
$$t \neq 1/2, \ t \neq 1 \ ,$$

$$A(t) = A_1(t) , \quad 0^+ \leq t < 1/2$$
$$A(t) = A_2(t) , \quad 1/2 < t < 1 ,$$

with the boundary conditions:

$$\Phi(1/2^+) = \Phi(1/2^-) , \quad \Phi(1^+) = \Phi(1^-) . \tag{3.10}$$

The numerical solution $\Phi(1) = C$ may be obtained within adequate accuracy by various techniques of numerical analysis. We shall apply the method of mean coefficients as outlined in the previous chapter (Ch. II, Sec. B-3). The fundamental period is divided into 21 subintervals in this example. The result is given by Eq. (3.11).

$$\Phi(1^+) = C \doteq \prod_{i=0}^{20} \exp A_i T_i , \tag{3.11}$$

where $\sum_{i=0}^{20} T_i = 1$.

The computational algorithms for calculation of the matrix C and its eigenvalues are available as standard computer routines in most computer laboratories.

b. Stability Analysis

In order to compute the matrix C and its eigenvalues, we choose the parameter values w_1, w_2, w_3, w_4 and Kb, and the period, as shown in Fig. 13. The characteristic roots and the characteristic exponents for these parameter values are plotted in Fig. 13 and Fig. 14a, respectively. It is interesting to compare the modulated feedback system shown in Fig. 10 with the unmodulated, continuous system which is obtained by setting $m_1(t) = m_2(t) = 1$ in Figs. 10 and 11. For the same choice of parameter values, the continuous system becomes unstable for the loop gain $Kb > 32.5$, as shown in Fig. 14b.

3. <u>Experimental Results</u>

The carrier frequency feedback amplifier shown in Fig. 10 is a "low frequency" model of the experimental amplifier shown in Fig. 15.

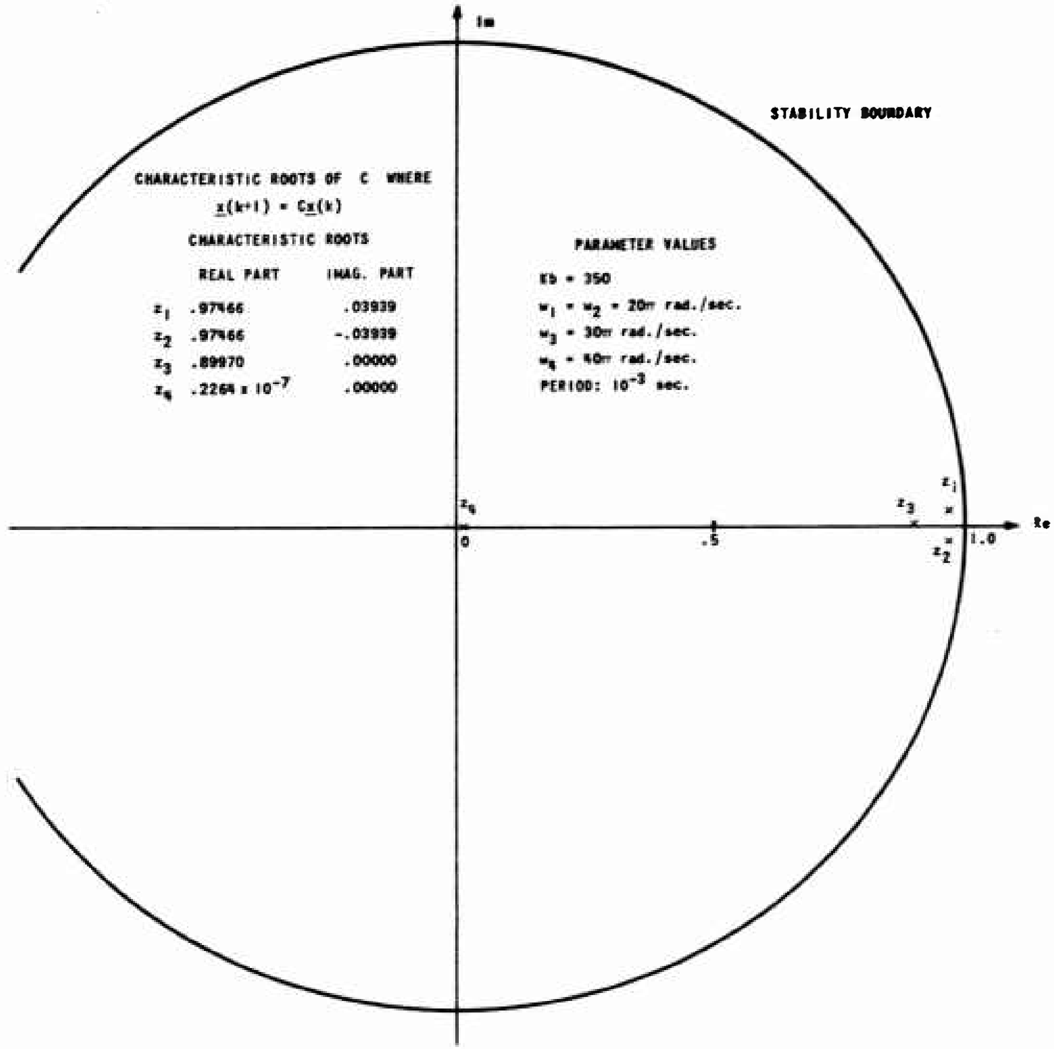
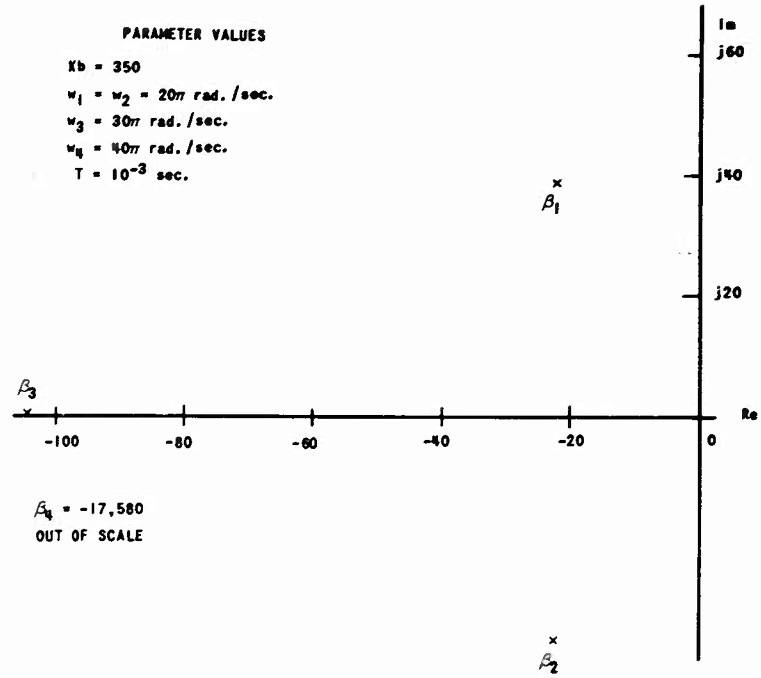
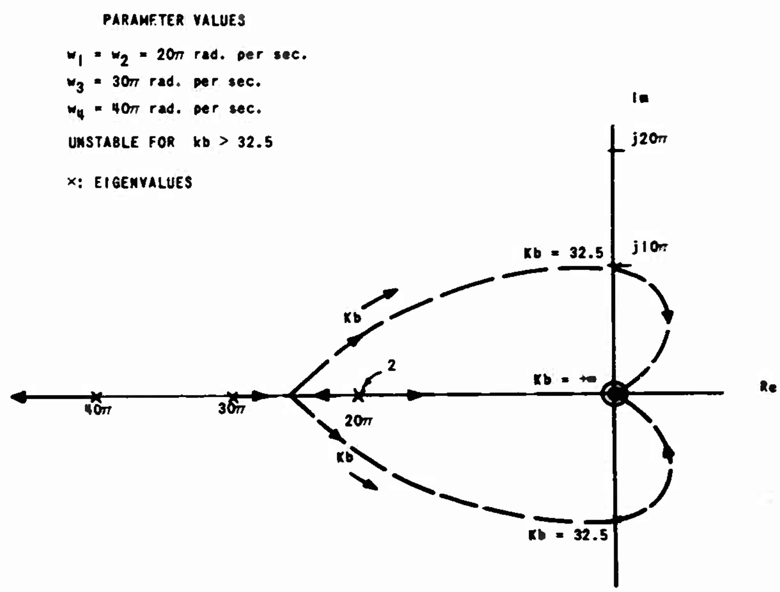


FIG. 13. Z-PLANE PLOT OF THE CHARACTERISTIC ROOTS OF THE MATRIX C.



a. S-Plane Plot of the Characteristic Exponents of the Matrix C.



b. S-Plane Locus of the Natural Frequencies of the Unmodulated, Continuous System.

FIG. 14. A COMPARISON OF THE STABILITY OF THE MODULATED SYSTEM WITH THAT OF AN UNMODULATED SYSTEM.

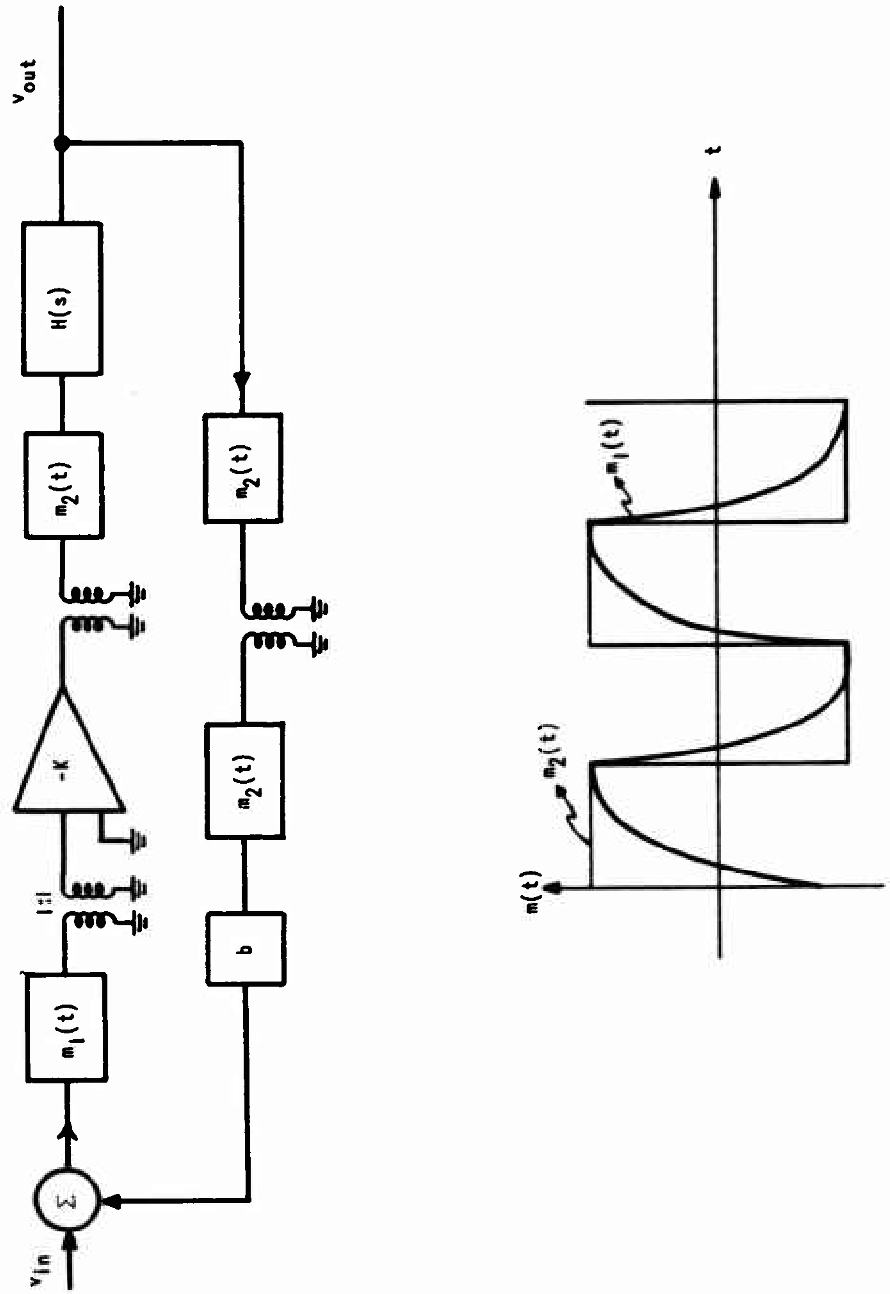


FIG. 15. EXPERIMENTAL AMPLIFIER

Since this is a D.C. amplifier, the frequency range of the input signal is limited to about 300 cps, and consequently the low frequency behavior of the system is of paramount importance. The transformers are used to block the transmission of any zero drifts associated with physical amplifiers and modulators, and to provide a better inter-stage isolation. The amplifier K and all three transformers are wideband, as shown in Figs. 16 and 17, and their upper cut-off frequencies are at least one order of magnitude greater than both the highest input frequency and the fundamental carrier frequency. Because of the wideband characteristics of the amplifier and transformers, the bandwidth of the loop transmission is predominantly determined by the cut-off frequency of the low-pass filter $H(s)$. This cut-off frequency is at least two orders of magnitude below the upper cut-off frequencies of the amplifier and transformers. The low frequency model of this amplifier is derived on the basis of the above reasoning. The amplifier is driven by the low duty factor, one cycle per second pulse train as shown in Fig. 18b. The system, Fig. 18a, is excited periodically for about .15 second and the input is zero for about .85 second. A natural decay of the system during the time of zero input is analyzed from the oscillograms in Fig. 19.

We know from the Floquet theory that any natural decay of the system must be expressible as a linear combination of the normal modes, and each normal mode is of the form $\underline{p}_i(t)e^{\beta_i t}$. Normally, we can measure one component of the state vector,

$$x_1(t) = \sum_{i=1}^{4} q_i(t)e^{\beta_i t}, \quad \text{where} \quad q_i(t+T) = q_i(t), \quad T = 10^{-3} \text{ sec.} \tag{3.12}$$

If $q_i(t)$ are expanded in Fourier series, then we have

$$q_i(t) = \sum_{n=-\infty}^{\infty} c_{in}e^{jn\Omega t}, \quad \text{where} \quad \Omega = 2\pi \times 10^3 \text{ rad. per sec.}$$

Passing $x_1(t)$ through the low-pass filter as shown in Fig. 18a, we

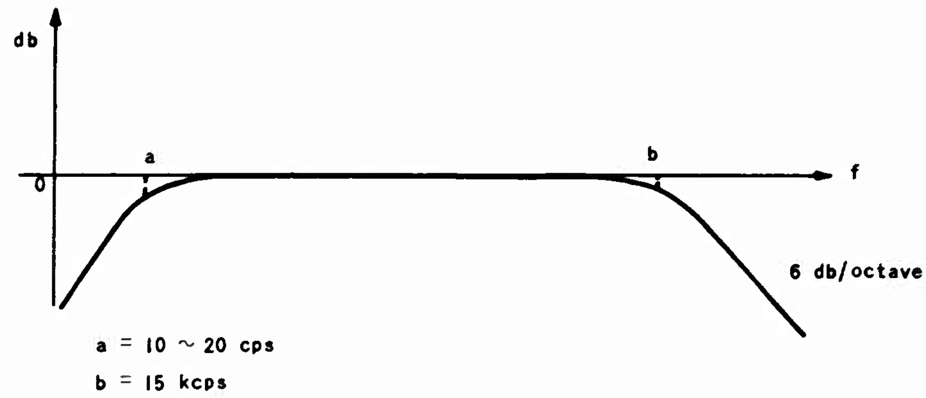


FIG. 16. AMPLITUDE CHARACTERISTICS OF TRANSFORMER.

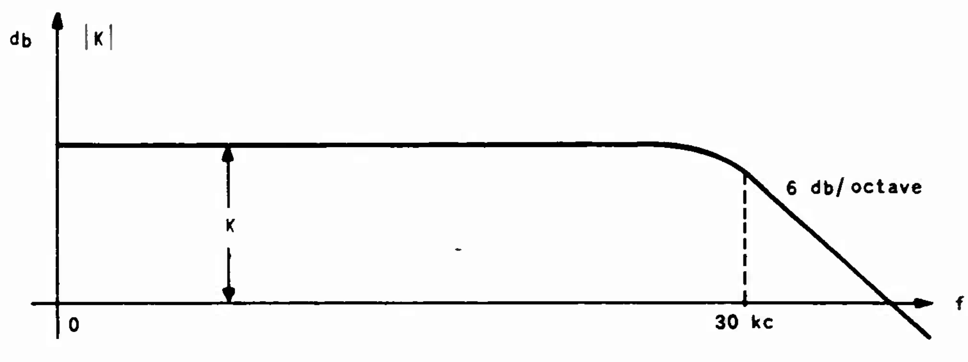
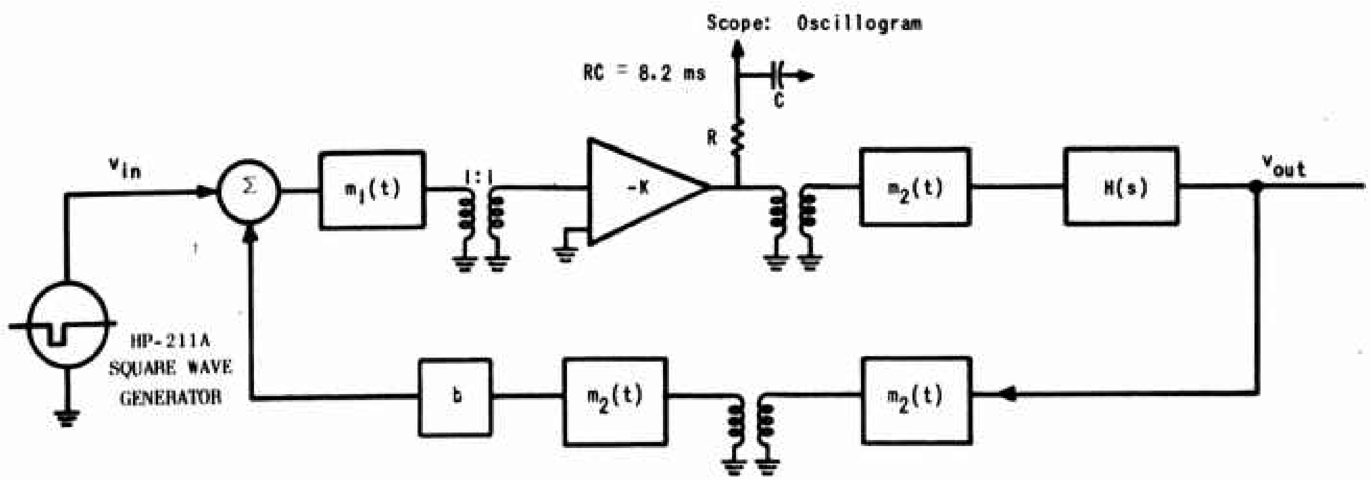
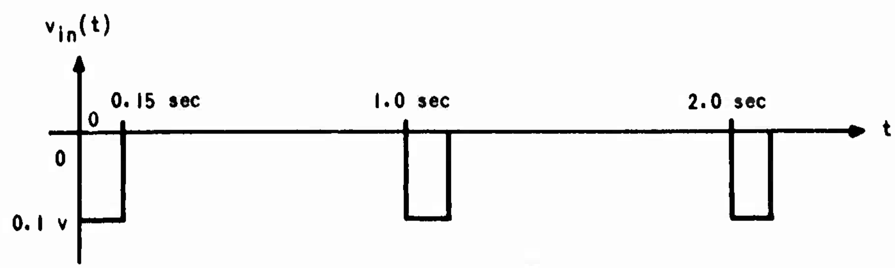


FIG. 17. AMPLITUDE CHARACTERISTICS OF AMPLIFIER.

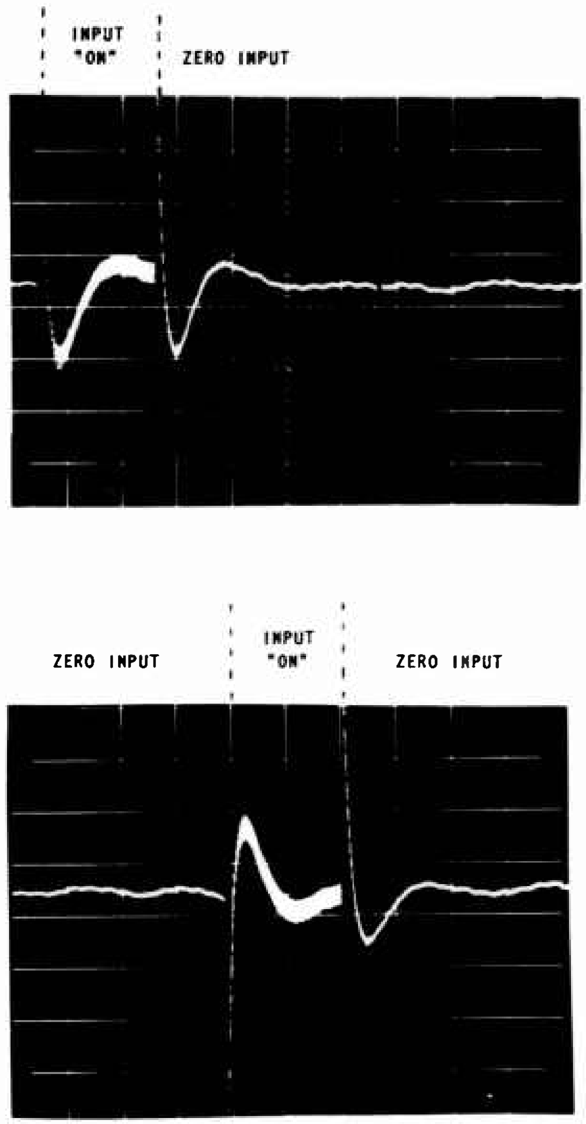


a. Experimental Setup.



b. The Input Waveform for Fig. 18a.

FIG. 18. EXPERIMENTAL METHOD OF DETERMINING THE NATURAL DECAY OF THE MODULATED SYSTEM.



PARAMETER VALUES

$K_b = 350$

$w_1 = w_2 = 20\pi$ rad./sec.

$w_3 = 30\pi$ rad./sec.

$w_4 = 40\pi$ rad./sec.

$T = 10^{-3}$ sec.

HORIZONTAL SCALE 50 msec/cm
VERTICAL SCALE 20 mv/cm

FIG. 19. RESPONSE OF CARRIER AMPLIFIER (FIG. 18a) TO .15 SECOND PULSE.

obtain the oscillograms of Fig. 19. Since all the terms of $q_1(t)$ except the constant terms are attenuated by the low-pass filter, only the sum of two dominant modes, $a_1 e^{\beta_1 t} + a_2 e^{\beta_2 t}$, is significant after about the first 50 milliseconds of decay. One can measure the exponents β_1 and β_2 approximately from the oscillogram. The measured and calculated values agree to within the limits of experimental accuracy, as shown below.

a. Natural Frequency of Oscillation

Calculated	7.5 cps
Measured	8.24 cps
% error	9.85 %

b. The Constant of Decay

Calculated	42 msec
Measured	30-40 msec .

C. LIMITED DESIGN BASED ON THE FLOQUET THEORY AND EXPERIMENTAL RESULTS

1. The Nature of Design Problems

There is a close relationship between analysis and design of any engineering system, and normally the development of simple, approximate methods of analysis forms an essential part of the design. The Bode diagram, Nyquist plot and Evans' root-locus method are the best known tools for engineers designing feedback control systems. The popularity of these methods may be attributed to the fact that they allow "workable" approximations and simple analysis of the system response as a function of the critical design parameters, such as loop gain or bandwidth.

The main difficulty in the design of linear systems with periodic parameters comes from the fact that we still do not have any simple method of analysis because we cannot in general obtain simple analytical relations between the characteristic roots and the system parameters. So long as we have no clear-cut insight into this relationship, we probably will not be able to develop powerful yet simple design methods for linear systems with periodic parameters to the same extent we have developed such methods for stationary linear systems. The actual design

of any engineering system can seldom be accomplished by pure synthesis in most practical situations. The system design may be divided into four steps, as follows [Ref. 36; also Ref. 1, pp. 36-40]:

The first step is to write the design specifications outlining the specific functions to be performed by the system and the desired qualities of its performance in terms of the degree of accuracy, the speed of response, the allowable peak overshoot, the signal-to-noise ratio, the power capability of the output stage and the estimated cost, etc.

The second step is to interpret as many of these specifications as possible in terms of constraints on the design parameters of the system. This is often the most difficult part of the system design. Normally the designer is able to translate only a part of the specifications into analytical constraints on the mathematical equations representing the system dynamics. Because of this difficulty, the actual design of any engineering system can seldom, if ever, be reduced to a pure mathematical synthesis.

The third step is to select the physical components, such as the compensating networks and the controller, in addition to the fixed parts of the system, to realize the mathematical functions derived from the manipulations of the system equations formulated in the second step.

The fourth and final step is to evaluate the design and to improve the performance by a combination of analog and digital simulations. Also included in this last step is the experimental testing of the laboratory models.

In most cases we can solve only parts of the design problem, although these may often be the most important parts, by analytical methods. In this paper we shall first consider the limited design of carrier-frequency feedback systems with general periodic modulations.

2. Extension of the Root-Locus Concept to the Design

The original root-locus method, as proposed by W. R. Evans, [Ref. 37], is a combination of analytical and graphical techniques that allows a rapid, approximate determination of the system eigenvalues (natural frequencies) as they are affected by variation of a critical design parameter, such as system loop gain. Normally the characteristic

equation of a single loop feedback system has the form

$$D(s) + KN(s) = 0 , \tag{3.13}$$

where $D(s)$ and $N(s)$ are polynomials in the complex variable s, and K is the variable parameter. A considerable simplification results if the polynomials $D(s)$ and $N(s)$ are known in factored forms; this is the case in which the root-locus method is most effective. If the polynomials $D(s)$ and $N(s)$ are not given in factored form, it is not possible to take full advantage of the root-locus method in determining the loci of the roots of Eq. (3.13) as K is varied over a certain range. If the critical parameter K does not appear in a simple form in the characteristic equation as shown by Eq. (3.13), it is almost hopeless to find a simple way to sketch the loci of characteristic roots as functions of the parameter K. Unfortunately this is the situation we find in a design of carrier-frequency feedback amplifiers. For example, in the linear system with piece-wise constant periodic parameters shown in Fig. 7, the characteristic roots are obtained from

$$\det [zI - C] = 0 ,$$

where $C = S_2 \exp A_2(1/2) S_1 \exp A_1(1/2)$, and where the design parameters are included in elements of the matrices A_1 and A_2.

The coefficients of this characteristic equation are very complicated transcendental functions of the design parameters, so that it is just about impossible to find useful, simple relationships between the characteristic roots and the design parameters. We are forced to use a computer to calculate the characteristic roots as one design parameter is varied at a time. For instance, we may vary the loop gain while holding the bandwidth to a certain value. We may then choose a suitable combination of the loop gain and bandwidth from the family of root loci plots obtained in this manner. This is not necessarily a simple, efficient method for designing a carrier-frequency feedback amplifier; but it is straightforward. It is a practical method if a digital computer is available.

Even though the root-locus method may be adequate for the adjustment of the design parameters, the use of this method is essentially design by repeated analysis, and hence is not applicable to the synthesis of compensating networks which are frequently needed in the feedback amplifier. This is perhaps the worst limitation of the design method discussed above.

In the next section we shall illustrate an application of the root-locus concept to the adjustment of design parameters.

3. A Design Example and Experimental Results

A number of examples of design by the root-locus concept, including design of one fifth-order system, have been worked out, and good agreement with experiment has been obtained. We shall take the carrier amplifier shown in Fig. 10 as a design example.

Normally there are four parts in the design specifications for a feedback amplifier. These are: gain, bandwidth, stability and noise. In the example based on Fig. 10, we assume that the random noises produced in amplifiers and modulators are negligible. We only consider the unwanted sideband components (modulation products) generated by the modulation.

a. Design Specifications

1. An external D.C. gain variable from 20 db to 60 db and constant within 2 % of the nominal design value.
2. Minimum bandwidth: 300 cps.
3. Dominant time constant: less than .05 sec. and $\zeta = .5$ (ζ=damping ratio).
4. Output signal to noise ratio: 60 db.
5. Fixed components of the system: the characteristics of modulators, transformers and amplifier as shown in Figs. 15, 16 and 17.

Usually there are additional design requirements not specified above, but we ignore them in this example.

Ideally, a designer should obtain the equations of the system with appropriate design constraints from the given specifications. The mathematical solutions of these equations then should give the set of parameter values required to realize the specified design. But the specifications given above are not completely reducible to such a set of purely logical and mathematical operations.

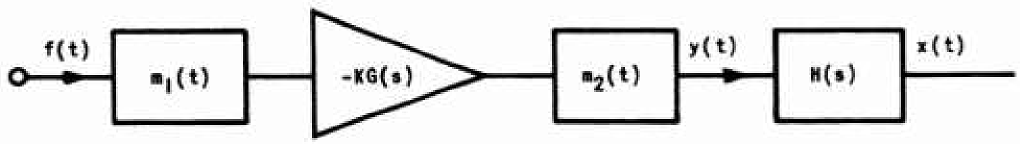
In the example now under consideration, we want to determine the cut-off frequency of a low-pass filter w_1, the gain constant K, and the feedback ratio b in such a manner as to meet the design specifications.

b. Preliminary Design Calculations

Now we actually calculate the preliminary design values of the system parameters. Normally in this phase of design, a designer uses simple approximate models to facilitate preliminary calculations of the design parameters. It is a matter of skill, experience, and physical intuition of the designer to make workable simple models appropriate to the design problems at hand. We shall use in this example a very simple and somewhat crude model based on frequency domain approximations for calculation of the D.C. gain and the minimum bandwidth. For the purpose of stability analysis, to obtain the specified time constant and damping ratio, we use the system model in Fig. 10 and apply the extended Floquet theory.

In order to derive the simple model suitable for calculation of the gain and bandwidth, we follow reasoning based on the frequency response characteristics of carrier systems.

First we consider the forward transmission link, which is redrawn in Fig. 20 for convenience. It is well known that the ultimate bandwidth of a carrier system such as shown above is limited by the fundamental carrier frequency Ω, and may never be greater than $\Omega/2$. We obtain the simple approximate model of the forward transmission link as shown in Fig. 21 if we assume:



$$G(s) = \frac{s^2}{(s+w_2)(s+w_3)}, \quad H(s) = \frac{w_1}{s+w_1}$$

FIG. 20. THE FORWARD TRANSMISSION LINK.

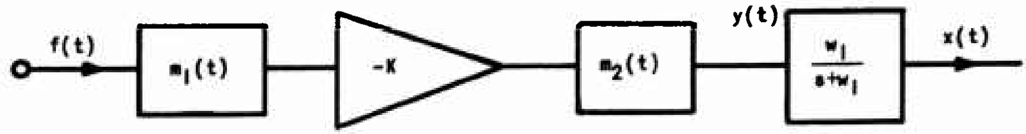


FIG. 21. A SIMPLIFIED MODEL OF THE FORWARD LINK.

1. the input signal is bandlimited to $\Omega/3$,
2. the total amount of energy in the sidebands is about 1% of the signal energy in the output.

In the example under consideration, the latter condition is generally satisfied for the range of the parameter values $0 \leq w_i \leq \Omega/10$, $i = 1, 2, 3$.

Finally, we obtain the simplified model in Fig. 22 for the complete system if the feedback link is also replaced by the same type of approximate model as used for the forward transmission link. The reduction of gain constant from K to $.7K$ is caused by the differences in phase and waveforms of $m_1(t)$ and $m_2(t)$. The average gain from $f(t)$ to $y(t)$ in Fig. 21 is about $-.7K$, which can readily be calculated from Fig. 11b.

Now we calculate the D.C. gain and bandwidth from the model in Fig. 22 for comparison with measured values from the experimental carrier amplifier in Fig. 15. The gain and bandwidth of a carrier system need to be defined carefully, because its output contains a countably infinite number of sidebands in addition to the signal-frequency

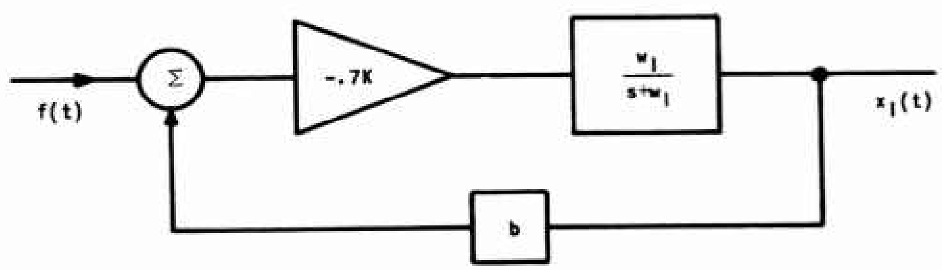


FIG. 22. THE SIMPLIFIED SYSTEM MODEL.

component. We therefore define the gain as the ratio of the amplitude of the signal-frequency component of the output to that of the input signal. We accordingly define the bandwidth as the frequency at which this gain falls 3 db below its D.C. value.

From the transfer function

$$T(s) = \frac{X_1(s)}{F(s)} = \frac{-.7Kw_1}{s + (1 + .7Kb)w_1} \tag{3.14}$$

it is an easy matter to calculate the gain and bandwidth. In order to maintain the constant gain within the specified tolerance of 2%, we set the loop gain $Kb = 100$, since the gain sensitivity is inversely proportional to the loop gain in the single loop feedback system.

The external gain K_o and the bandwidth B_w are

$$K_o = -\frac{1}{b}, \quad B_w = (1 + .7Kb)w_1, \tag{3.15}$$

and $10^{-3} \leq b \leq 10^{-1}$ with $10^3 \leq K \leq 10^5$ in order to vary K_o from 20db to 60db with the loop gain $Kb = 100$.

The bandwidth calculated from Eq. (3.15) and the measured values from the experimental amplifier shown in Fig. 15 are compared below:

1. $Kb = 100$, $w_1 = 8\pi$ rad./sec. ($\pm$ 10%)
   Calculated B_w = 284 cps ($\pm$10%)
   Measured B_w = 272 cps
   % error = 4.4% (-5.9% to +11.1%)

2. $Kb = 100$, $w_1 = 12\pi$ rad./sec. ($\pm$ 10%)
   Calculated B_w = 426 cps ($\pm$ 10%)
   Measured B_w = 365 cps
   % error = 16.7% (+5% to +28.2%),

where other parameters are: $w_2 = 20\pi$ rad. per sec., $w_3 = 30\pi$ rad. per sec. and $w_4 = 40\pi$ rad. per sec.

These data indicate that the errors in bandwidth calculations based on the simplified model increase as the input signal frequency

and the bandwidth B_w approach the ultimate limit, $\Omega/2$, which is equal to 500 cps in this example. The agreement between the calculated and measured external gain K_o, however, was well within the specified tolerance of $\pm$ 2%.

Having calculated the gain and bandwidth from the simplified model shown in Fig. 22, and having obtained experimental support for the model, one may be tempted to use the same model for stability analysis. Here one must be <u>warned</u> very strictly that the model under consideration is <u>not</u> <u>valid</u> for stability analysis, as will be shown in the next section.

c. Stability Analysis for the Design

It appears from Eq. (3.14) that the natural response remains well damped even if the loop gain Kb increases without bound. This conclusion which is based on the simplified model, is proved <u>completely wrong</u> by stability analysis based on the original model shown in Fig. 10, which is the more accurate model of the experimental amplifier shown in Fig. 15.

In order to find the allowable range of parameter values which satisfy the design requirements, we extend the basic concepts of Evans' root-locus method. Since the characteristic exponents determine the dominant time constant and the damping, we plot the migration of characteristic exponents as functions of the loop gain Kb in Fig. 23. Because of the extremely complicated relationships between the characteristic exponents and the loop gain, the numerical values of the characteristic exponents corresponding to different settings of the loop gain Kb are obtained by machine computation. The entire computation takes no more than a few minutes if a digital computer is used. In Fig. 23, all the system parameters except the loop gain are held constant. The shaded area is the "forbidden region". The characteristic exponents must be located to the left of this region to satisfy the design specifications. One may also plot the loci of characteristic exponents as functions of the low-pass cut-off frequency w_1 while holding all other system parameters constant. This is done in Fig. 24. From plots of this type, one could also construct a map in the two-dimensional parameter space, Kb-w_1 plane, to plot the boundary which

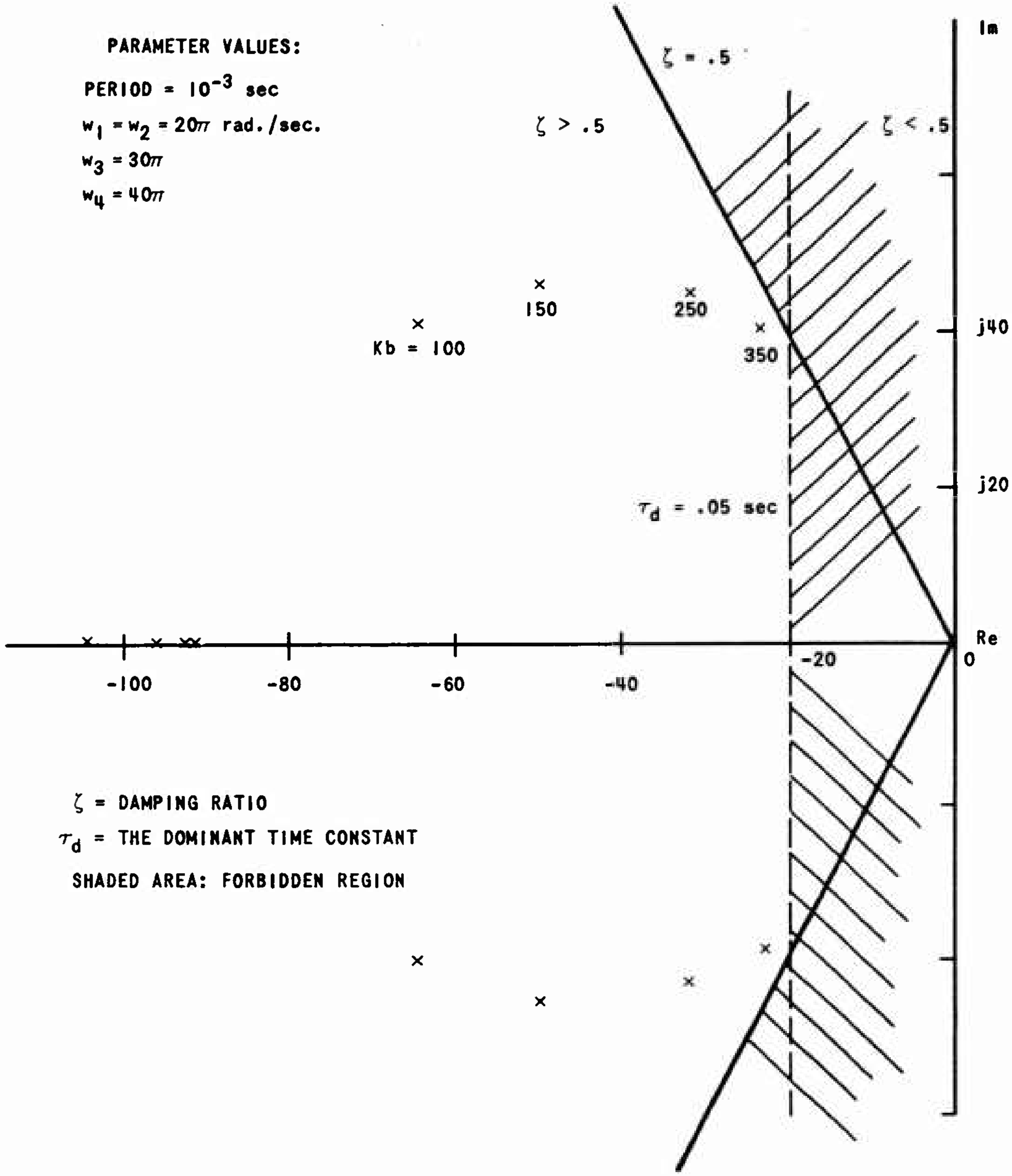


FIG. 23. THE LOCI OF CHARACTERISTIC EXPONENTS AS FUNCTIONS OF LOOP GAIN Kb.

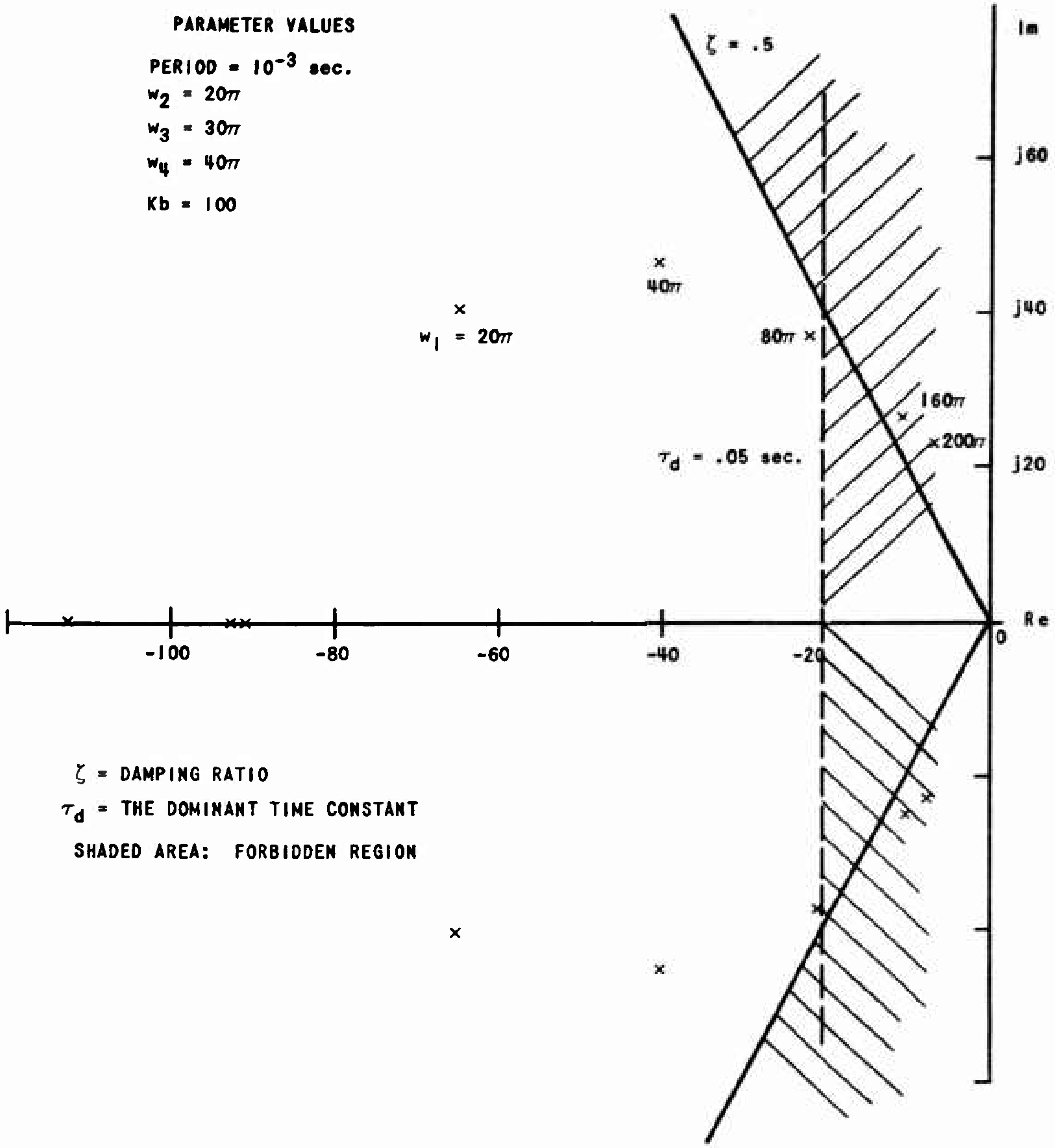


FIG. 24. THE LOCI OF CHARACTERISTIC EXPONENTS AS FUNCTIONS OF THE CUT-OFF FREQUENCY w_1.

separates the plane into the allowed and forbidden areas.

d. The Output Signal-to-Noise Ratio

The weakness of the present design method shows up most clearly at this point. If we choose the three parameters, K, b, and w_1 to meet the design specifications on the D.C. gain, bandwidth, dominant time constant, and damping ratio), then we have little choice but to accept whatever signal-to-noise ratio results from the above choice of design parameters. It is clear from the system model shown in Fig. 10 that the high frequency noises generated by the modulators are attenuated mostly by the low-pass filter. The cut-off frequency of this filter w_1 determines the bandwidth, as already shown by Eq. (3.15). It can provide, however, only fixed 20 db/decade attenuation rate. This is not enough to meet the specified output signal-to-noise ratio of 60 db.

Perhaps the simplest (but not necessarily the most elegant) solution is to filter the output $x_1(t)$ in an adequate manner outside the feedback loop, as shown in Fig. 25. The output filter $H_o(s)$ may be designed by a number of well-known network design techniques [Refs. 38, 39]. The fourth-order Butterworth filter is found to have adequate cut-off characteristics for this application. The signal-to-noise ratio in the filtered output $x_o(t)$ now meets the design specification. The element values of the filter may be found directly from available tables [Ref. 40].

e. Concluding Remarks on the Design

The design example discussed above shows that linear feed-back systems with piece-wise continuous periodic parameters may be

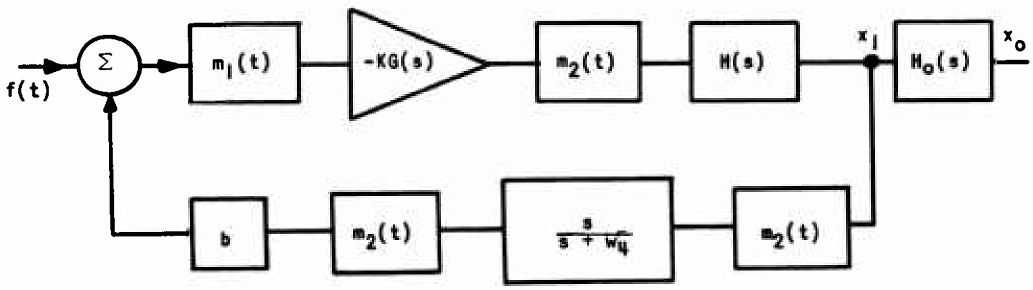


FIG. 25 A COMPLETE SYSTEM MODEL

designed in almost the same manner as we design stationary linear feedback systems. The familiar concepts and techniques such as frequency domain approximations and root-locus method apply directly to the design of linear systems with periodic parameters. The type of stability analysis and the filtering of high frequency modulation products are the only major differences between stationary linear systems and linear systems with periodic parameters. Because of the great degree of similarity between the two classes of linear systems, the design of linear systems with periodic parameters will be greatly facilitated if there are reliable systematic procedures by which one can always find the approximately equivalent stationary models adequate for analysis and design. It seems possible to develop such approximation procedures if the carrier to signal-frequency ratio is not lower than 10:1. This approximation problem is not studied in this chapter; but it seems important enough to deserve future investigations.

The problems concerning analytical design and synthesis of compensating networks have also not been discussed to this point. Since the Floquet theory is primarily a tool of analysis, there seems to be no straightforward way to apply it to the compensation of linear feedback systems with periodic parameters. This is clearly seen from the fact that one must have the differential equation of the compensated system to apply the Floquet theory, but one cannot write the differential equation unless one knows the structure of the compensating network. Once a specific form of compensation, such as a lead network or a lag network, is assumed, one can adjust parameter values of the compensating network in the same manner that the gain and bandwidth are adjusted in the design example. It is, however, not possible to obtain the structure and parameter values of a desired compensating network from the Floquet theory. Some aspects of these problems will be discussed in the next chapter, wherein we study the application of integral equations to the synthesis of compensating networks.

IV. APPLICATIONS OF INTEGRAL EQUATIONS TO THE STUDY OF LINEAR FEEDBACK SYSTEMS

A. REPRESENTATION OF LINEAR FEEDBACK SYSTEMS BY VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

In the study of physical systems which can be represented by differential equations, one may often gain a better insight into the physical nature of the problem, or one may solve the problem more easily, by representing the given physical system by an integral equation.

First it is shown in this chapter that a wide class of linear feedback systems (including stationary, sampled-data and carrier frequency feedback systems) may be studied from a unified point of view by applications of Volterra integral equations of the second kind.

Second, it is shown that the integral equation formulation leads to a straightforward mathematical solution of the compensation problem for a limited class of linear feedback systems with periodic parameters

1. Integral Equations as Generalizations of the Convolution Integral Approach

During the past two decades, analysis and design of time-invariant (stationary) linear systems have been carried out in terms of the convolution integral and associated convolution transforms, among which the Fourier transform and the Laplace transform are perhaps the best known methods. These transform methods are most effective in solving special classes of linear integral equations. We shall now consider two simple examples of linear feedback systems which naturally lead to linear integral equations.

Example (1)

The system equation for Fig. 26 is:

$$x(t) = \int_0^t e(\tau)h(t - \tau)d\tau = \int_0^t [f(\tau) + bx(\tau)]h(t - \tau)\, d\tau \ ,$$

or

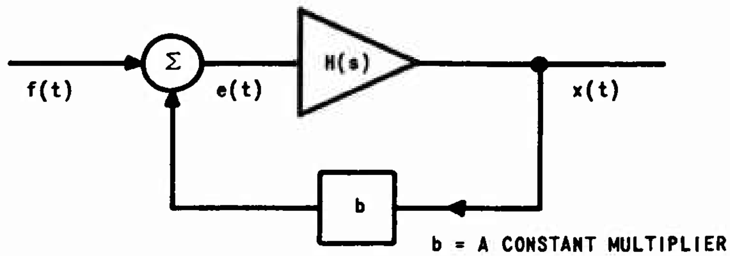


FIG. 26. A SIMPLE STATIONARY FEEDBACK SYSTEM (EXAMPLE 1).

$$x(t) - b\int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t f(\tau)h(t-\tau)d\tau \; , \qquad (4.1)$$

$$t \geq 0^+, \quad x(0^+) = 0.$$

The unknown, $x(t)$, appears under the integral sign. This is a convolution type integral equation, a special case of the Volterra integral equation of the second kind. (See Ref. 17, Chapter II, and Ref. 41, Chapter II.) Equation (4.1) can be reduced to the familiar algebraic equation by the Laplace transform:

$$X(s) - bX(s)H(s) = F(s)H(s) \; , \qquad (4.1a)$$

or

$$X(s) = \frac{H(s)}{1 - bH(s)} F(s) = W(s)F(s) \; .$$

This is normally regarded as the basic equation of the single loop feedback system in Fig. 26. Although the transfer function concept based on the above equation is very useful, this specialized approach has serious limitations if the system is no longer stationary (as will be shown next).

Example (2)

m(t) = A TIME-DEPENDENT MULTIPLIER

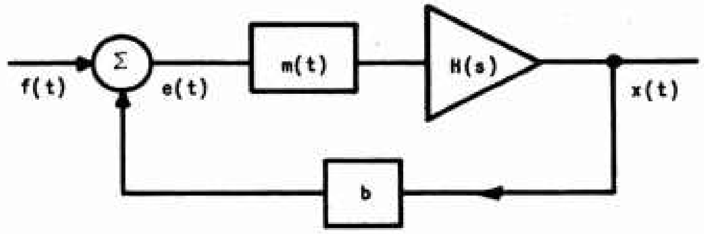


FIG. 27. A SIMPLE NONSTATIONARY FEEDBACK SYSTEM (EXAMPLE 2).

Figure 27 represents a simple example of nonstationary feedback systems with a modulator $m(t)$ and the transfer function $H(s)$. The system equation is:

$$x(t) - b\int_0^t x(\tau)m(\tau)h(t - \tau)d\tau = y(t) \ , \ t \geq 0^+ \qquad (4.2)$$

where $y(t) = \int_0^t f(\tau)m(\tau)h(t - \tau)d\tau$ is the "open loop" response when there is no feedback, i.e., when $b = 0$.

Taking the Laplace transform, we have

$$X(s) - [X(s)*M(s)]H(s) = Y(s) \ , \qquad (4.3)$$

where $*$ denotes the convolution in s.
This is not an algebraic equation in $X(s)$ but a convolution type integral equation in the complex variable s. It is not possible to solve this equation analytically except in special cases; consequently we cannot calculate the transfer function as in the stationary case. Even for the simplest possible modulation, $m(t) = 2 \cos wt$, Eq. (4.3) becomes

$$X(s) - [X(s - jw) + X(s + jw)]H(s) = Y(s) \ . \qquad (4.4)$$

This is a functional equation in the complex variable s. The mathe-

matical theory on this subject does not seem to be adequately developed. Even a very simple example such as the one now under consideration is enough to point out the rather serious limitations in the application of the Laplace transform method to the analysis of nonstationary linear systems. It is therefore not only desirable, but necessary, to start with more basic concepts than those of transfer function and Laplace transform in order to formulate a more general theory applicable to both stationary and nonstationary linear systems.

The Volterra integral equation of the second kind seems to provide an excellent tool for analytical investigations of general linear systems. A standard form of this equation is

$$x(t) - b\int_0^t k(t, \tau)x(\tau)d\tau = y(t) , \qquad (4.5)$$

where $y(t)$ and $k(t, \tau)$ are known, and $k(t, \tau)$ is called the kernel. A physical system represented by Eq. (4.5) is the time-varying linear feedback system shown in Fig. 28, if we set

$$y(t) = \int_0^t f(\tau)k(t, \tau)d\tau .$$

Equation (4.5) reduces to Eq. (4.1) if the system is stationary; i.e., if $k(t, \tau) = h(t - \tau)$.

In the case of many modulation systems, the kernel has the following form:

$$k(t, \tau) = m(\tau)h(t - \tau) . \qquad (4.6)$$

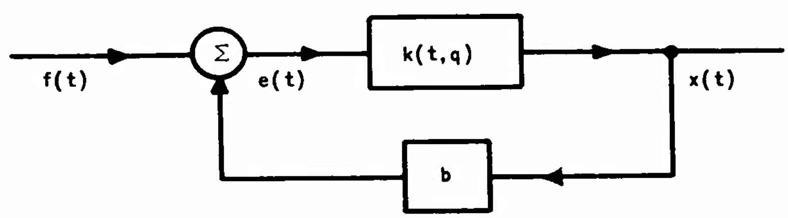


FIG. 28. A PHYSICAL MODEL OF THE VOLTERRA EQUATION.

This is called a separable kernel [Ref. 42], or product-type kernel, and plays a very important role in the study of periodically sampled systems and amplitude modulation systems.

In general it is not possible to tell the structure of the physical system from the differential equation of the system alone, because there are many different physical systems which lead to the same form of differential equation. For instance, the form of a differential equation describing a feedback system is not different from the form of a differential equation describing a stable system without any intentional feedback. Unless we have *a priori* knowledge that the given differential equation comes from a stable system, the form of the equation, $\dot{\underline{x}} = A(t)\underline{x}$, does not reveal whether the system is stable or not.

In the integral equation representation, on the other hand, it is possible to indicate clearly the presence of any intentional feedback in the system. Furthermore, it is more convenient to talk about the general properties of the system in the integral equation formulation than in the differential equation formulation. The relative merits of the two formulations will become clearer as we go along.

2. Conversion of the Ordinary Differential Equation into a Volterra Integral Equation

Now we set up a general procedure to be followed in going from one representation of the physical system, namely, the ordinary linear differential equation, to the other representation, that is, the Volterra integral equation of the second kind. The theory of Volterra integral equations of the second kind is particularly simple, and the estimates on the bounds of the solutions can be obtained more readily than in the differential equation formulation of the same problem.

Consider the following differential equation:

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + \underline{u}(t) \ , \quad \underline{x}(0) = \underline{0} \ , \ 0 \le t < \infty \ , \qquad (4.7)$$

$A(t)$: continuous and bounded $n \times n$ matrix ,

$\underline{u}(t)$: continuous and bounded $n \times 1$ matrix .

The formal solution of this equation as given by Eq. (2.19) is of little use in practice because we cannot find in general the fundamental matrix $\Phi(t)$. If we convert, however, the above equation into a Volterra integral equation of the second kind [Ref. 33, pp. I-11 to I-13], then we can find an upper bound and a lower bound of the solution $\underline{x}(t)$ readily by application of Gronwall's lemma [Ref. 12, p. 35, and Ref. 27, pp. 40-41] and the theory of linear operators [Ref. 29, Chapters 3 and 4; Ref. 43, pp. 48-93; and Ref. 44].

In order to convert Eq. (4.7) into a Volterra integral equation of the second kind, we decompose the time-dependent matrix $A(t)$ into a sum of two matrices: a simple constant matrix W and a time-dependent matrix $B(t)$.

$$A(t) = W + B(t) \quad . \tag{4.7a}$$

This decomposition is arbitrary, except that the constant matrix W should be so simple that its exponential, $\exp Wt$, can be analytically calculated without tedious labor.

The new differential equation becomes

$$\dot{\underline{x}}(t) = W\underline{x}(t) + B(t)\underline{x}(t) + \underline{u}(t) \quad . \tag{4.8}$$

This can be transformed into a Volterra integral equation of the second kind:

$$\underline{x}(t) - \int_0^t H(t - \tau)B(\tau)\underline{x}(\tau)d\tau = \int_0^t H(t - \tau)\underline{u}(\tau)d\tau \ , \tag{4.9}$$

where $H(t) = \exp Wt$.

This is the vector-matrix form of the scalar equation (4.1) and physically corresponds to the multi-dimensional feedback system shown in Fig. 29.

Equation (4.9) may be written in a simpler form with the aid of operator notation.

$$\underline{x} - K B\underline{x} = K\underline{u} \quad \text{or} \quad [I - KB]\underline{x} = K\underline{u} \ , \tag{4.9a}$$

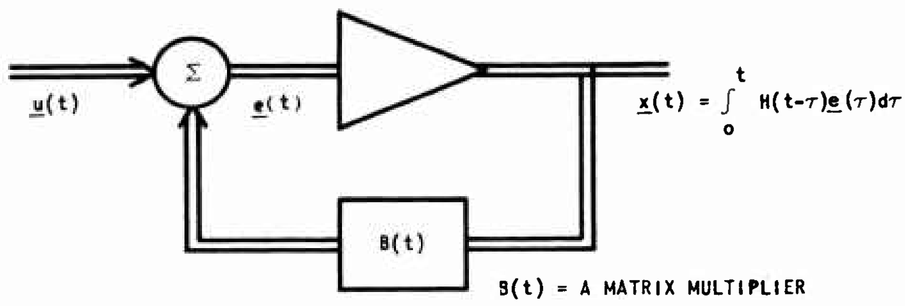


FIG. 29. A MULTI-DIMENSIONAL FEEDBACK SYSTEM.

where K is the convolution operator defined by $K\underline{x} \equiv \int_0^t H(t - \tau)\underline{x}(\tau)d\tau$. This is a linear operator equation. Now we can apply the theory of linear operators to the above equation to solve for $\underline{x}$ and find its bounds, as will be shown in the next sections.

B. ELEMENTS OF INTEGRAL EQUATION THEORY

We have given examples of the representation of linear dynamic systems by integral equations. Now we discuss a limited aspect of integral equation theory which is directly applicable to the study of linear feedback systems.

1. The Classification of Linear Integral Equations

Linear integral equations are basically classified into three kinds. We list each kind below and point out the physical significance wherever possible.

a. Linear Fredholm integral equation of the first kind:

$$\int_a^b h(t, \tau)x(\tau)d\tau = f(t), \qquad a \leq t \leq b \, , \tag{4.11}$$

b. Linear Fredholm integral equation of the second kind:

$$x(t) - \lambda \int_a^b h(t, \tau)x(\tau)d\tau = f(t) \, . \tag{4.12}$$

where λ = a parameter.

c. Linear Volterra integral equation of the first kind:

$$\int_a^t h(t, \tau)x(\tau)d\tau = f(t), \qquad a \le t < \infty \tag{4.13}$$

d. Linear Volterra integral equation of the second kind:

$$x(t) - \lambda \int_a^t h(t, \tau)x(\tau)d\tau = f(t) , \tag{4.14}$$

e. Linear integral equation of the third kind:

$$g(t)x(t) - \lambda \int_a^b h(t, \tau)x(\tau)d\tau = f(t) . \tag{4.15}$$

The unknown $x(t)$ always appears under the integral sign, and all other functions in the above equations are known. The function $h(t, \tau)$ is called the kernel of the integral equation. The Fredholm equations have the fixed limits of integration, a and b, while the Volterra equations have the variable upper limit, t.

The well-known Wiener-Hopf equation for the nonstationary random process [Ref. 45] is a special case of Eq. (4.11). Equation (4.14) is the same as Eq. (4.5), which represents the time-varying feedback system in Fig. 28.

2. The General Method of Solution

The solution of the inhomogeneous differential equation (4.7) has the form

$$\underline{x}(t) = \int_0^t \Phi(t)\Phi^{-1}(\tau)\underline{u}(\tau)d\tau .$$

But this equation is of little use, as mentioned previously, because in practice we cannot generally find the fundamental matrix $\Phi(t)$ explicitly.

The situation seems somewhat improved, at least in principle, if the differential equation is converted into the Volterra integral

equation of the second kind; because it is then always possible to write down the solution of this integral equation explicitly by the Neumann series expansion [Ref. 31, pp. 34-37, and Ref. 46]. Since each term of the series is an integral, it is necessary to evaluate these integrals in order to put the solution in a more useful form.

Now let us consider a Volterra integral equation of the second kind in vector form,

$$\underline{x}(t) - \int_0^t H(t, \tau)\underline{x}(\tau)d\tau = \underline{f}(t) \,, \tag{4.16}$$

where $\underline{f}(t)$ is bounded for all $t \geq 0^+$.
The solution of this equation (see Ref. 41, pp. 13-15, and Ref. 47) is:

$$\begin{aligned} \underline{x}(t) = \underline{f}(t) &+ \int_0^t H(t, \tau)\underline{f}(\tau)d\tau + \int_0^t H(t, \tau)\underline{f}_1(\tau)d\tau + \ldots \\ &+ \int_0^t H(t, \tau)\underline{f}_n(\tau)d\tau + \ldots \end{aligned} \tag{4.17}$$

where $\underline{f}_1(t) = \int_0^t H(t, \tau)\underline{f}(\tau)d\tau$, and $\underline{f}_n(t) = \int_0^t H(t, \tau)\underline{f}_{n-1}(\tau)d\tau$.
This solution is valid only if the series converges uniformly.

It is necessary and sufficient for the uniform convergence that the norm of $H(t, \tau)$ be bounded in the closed region, $0^+ \leq t \leq T$, $0^+ \leq \tau \leq t$, for any finite T.

It is sufficient for the uniform convergence, but not necessary that $H(t, \tau)$ be bounded and <u>continuous</u> with respect to t and τ.

The solution $\underline{x}(t)$ is continuous if the kernel $H(t, \tau)$ is <u>either</u> continuous with respect to t and τ <u>or</u> its discontinuities are limited to the types specified by Kolmogorov and Fomin [Ref. 29, pp. 112-116]. From the latter case, one can infer a precise condition under which the solution of a linear differential equation with piece-wise continuous coefficients may be continuous. This is done by transforming the differential equation into the Volterra integral equation of the second kind with a piece-wise continuous kernel. Previously, in Chapter II, we have stated the existence of continuous solutions for piece-wise continuous, linear periodic systems on the basis of

physical reasoning. The mathematical condition is now given in the reference cited above.

The solution given by the Neumann series, Eq. (4.17), is of little practical value unless the integrals can be evaluated readily and unless the series converges very rapidly. Unfortunately, such is not generally the case in practice, and one must normally find some approximate methods of estimating the infinite sum of terms. Because of this fact, the Neumann series solution, Eq.(4.17), is primarily of theoretical interest and is not suitable to practical applications except in special cases.

C. ANALYSIS AND COMPENSATION OF FEEDBACK SYSTEMS BY VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

1. Introduction

It has already been shown that general, nonstationary linear feedback systems may be represented by Volterra integral equations of the second kind. Now we want to explore the advantages of integral equation representations in the analysis and compensation of a very broad class of feedback systems. It is shown hereafter that stationary linear systems, periodically sampled systems and nonstationary linear systems all may be represented by a single integral equation. Each class of the linear systems mentioned above corresponds to special conditions on the kernel of an integral equation. We can deduce, therefore, the known classes of linear feedback systems simply from the Volterra integral equation of the second kind, and thus place the specialized tools such as Laplace transform and Z-transform in a clear perspective.

The synthesis of compensating networks for a limited class of nonstationary feedback systems is most readily formulated by means of the integral equation. This feature seems to be one of the potential advantages of the integral equation representation over the differential equation representation.

2. The Stationary Linear Feedback System

We derive the stationary linear feedback system as a special case of the nonstationary system represented by Eq. (4.9) and sketched

in Fig. 29.

$$\underline{x}(t) - \int_0^t H(t - \tau)\ B(\tau)\underline{x}(\tau)d\tau = \int_0^t H(t - \tau)\underline{u}(\tau)d\tau\ .$$

Since the time-dependent matrix $B(t)$ represents the nonstationary part of the system, as may be seen clearly from Eq. (4.8), we set $B(t) = B_o$, a constant $n \times n$ matrix. Then we obtain the stationary system

$$\underline{x}(t) - \int_0^t H(t - \tau)B_o\underline{x}(\tau)d\tau = \int_0^t H(t - \tau)\underline{u}(\tau)d\tau\ . \tag{4.18}$$

Taking the Laplace transform of both sides, we have

$$\underline{x}(s) - H(s)B_o\underline{x}(s) = [I - H(s)B_o]\underline{x}(s) = H(s)\underline{u}(s)\ , \tag{4.19}$$

and

$$\underline{x}(s) = [I - H(s)B_o]^{-1}H(s)\underline{u}(s)\ .$$

From this equation, we can now define the generalized transfer function as

$$T_{ij}(s) = \frac{x_j(s)}{u_i(s)}\ . \tag{4.20}$$

This coincides with the conventional transfer function defined for the single-variable system, if x, u, H and B_o are all scalars in Eq. (4.18). We note that it is also possible to derive the same result from the differential equation (4.8) if we set $B(t) = B_o$. An interested reader may consult the article by Kavanagh [Ref. 48] for an extensive treatment of multi-variable control systems.

3. The Sampled-Data Feedback System

We may regard the simple nonstationary feedback system shown in Fig. 27 as the basic model of sampled-data systems if the modulator, $m(t)$, is the sampler defined by

$$m(t) = \sum_{k=-\infty}^{\infty} \delta(t - k) \equiv i(t)\ , \tag{4.21}$$

where k is an integer and $\delta(t)$ is the impulse. In order to derive the conventional result, we use Eq. (4.2), which is a scalar form of Eq. (4.9).

$$x(t) - b\int_0^t x(\tau)i(\tau)h(t - \tau)d\tau = \int_0^t f(\tau)i(\tau)h(t - \tau)d\tau \ . \tag{4.22}$$

If we take the Laplace transform of both sides,

$$X(s) - b[X(s)*I(s)]H(s) = [F(s)*I(s)]H(s) \ . \tag{4.23}$$

This is the same type of integral equation as Eq. (4.3), but the following important property of $I(s)$ makes it possible to solve this equation [Ref. 1, pp. 109-110]:

$$I(s)*\left\{[F(s)*I(s)]H(s)\right\} = [F(s)*I(s)][H(s)*I(s)] \tag{4.24}$$

In order to simplify the notation, we shall henceforth set $X(s)*I(s) \equiv X^*(s)$. If we convolute $I(s)$ with Eq. (4.23), we obtain

$$X^*(s) - bX^*(s)H^*(s) = F^*(s)H^*(s) \ , \tag{4.25}$$

and

$$X^*(s) = \frac{H^*(s)}{1 - bH^*(s)} F^*(s) = T^*(s)F^*(s) \ .$$

One may introduce the conventional Z-transform here to express the above result in the Z-domain:

$$X(z) = \frac{H(z)}{1 - bH(z)} F(z) = T(z)F(z) \ . \tag{4.26}$$

It should be clearly understood that the solution of Eq. (4.23) was possible only because of the special property of $I(s)$ expressed by Eq. (4.24). Otherwise, it is generally not possible to solve the integral equations (4.22) or (4.23) explicitly in a closed form.

We also note that it is not necessary to introduce the de-sampler

or holding device, as is done in many cases [Refs. 6, 7], when the sampled-data system is represented by differential equations. If the de-sampler is not allowed, then the differential equation of a sampled-data system will contain at least impulses in its coefficients. This can be seen from Eq. (3.3), if we replace the modulators $m_1(t)$ and $m_2(t)$ by appropriate impulse samplers $i_1(t)$ and $i_2(t)$. Although the differential equation with distributional coefficients may be handled by the theory of distributions [Ref. 31, pp. 135-143], the integral equation provides a more elegant and simpler treatment of the sampled-data system.

4. Estimates on the Bounds of Solutions

It is quite convenient to estimate the lower bound of the norm of the state vector in the integral equation representation. In many cases, this lower bound corresponds to the steady state "gain" of the system. For instance let us consider the multi-dimensional feedback system shown in Fig. 29. Using the simpler form of the system equation

$$\underline{x} - K B \underline{x} = K \underline{u} \ ,$$

we obtain the following inequality from the property of the norm [Ref. 43, pp. 81-83]:

$$\|\underline{x}\| \leq \frac{\|K\underline{u}\|}{1 + \|KB\|} \leq \frac{\left\| \int_0^t H(t-\tau)\underline{u}(\tau)d\tau \right\|}{1 + \int_0^t \|H(t-\tau)\| \, \|B(\tau)\| \, d\tau} \ . \tag{4.27}$$

It is also possible to derive from this equation the result analogous to the familiar feedback gain formula

$$\|\underline{x}\| \doteq \frac{\|K\|}{1 + \|K\| \, \|B\|} \|\underline{u}\| \doteq \frac{1}{\|B\|} \|\underline{u}\| \tag{4.28}$$

if $\|K\| \, \|B\| \gg 1$.

It is assumed in the above that $\underline{x}$ is bounded. The application of Eq. (4.28) to the 4th order system shown in Fig. 10 has shown the same

result that was obtained by the frequency domain approximation discussed in the previous chapter.

The equations (4.27) and (4.28) give estimates of "magnitude" of the system response, but neither gives any indication of stability. In order to obtain useful stability bounds, it is necessary to estimate an upper bound of the system response $\|\underline{x}\|$ as a function of critical system parameters such as gain and bandwidth, etc. Unfortunately, estimates of upper bounds based on the property of the norm

$$\|\underline{x}\| \leq \frac{\|K\underline{u}\|}{1 - \|KB\|}, \tag{4.29}$$

or on the Gronwall's lemma, give too conservative results to be useful for design of feedback systems. For example, if we apply Eq. (4.29) or Gronwall's lemma to Example (1) shown in Chapter II, we obtain the extremely conservative result from both estimates that the system is stable for $KbM < b$. But it is seen from the exact solution of the system equation (2.24) that the system could be stable even if $KbM \to \infty$.

It may be tentatively concluded from the above discussion that the estimates of stability bounds based on the norm are too crude and too conservative to be useful for practical applications. It is, therefore, necessary to develop more refined estimates of stability bounds for successful applications of the integral equation representation to analysis and design of nonstationary linear feedback systems. So far, no satisfactory mathematical theory on stability of the Volterra integral equations has emerged.

Next we shall consider application of integral equations to the synthesis of a limited class of nonstationary feedback systems.

5. Compensation of Nonstationary Linear Feedback Systems

We now consider a class of nonstationary linear systems which may be separated into stationary parts characterized by transfer functions and time-dependent multipliers which may represent modulators or time-variable gain.

The feedback system shown in Fig. 30 belongs to this class. We assume that the input $f(t)$, the response $x(t)$ and all the components

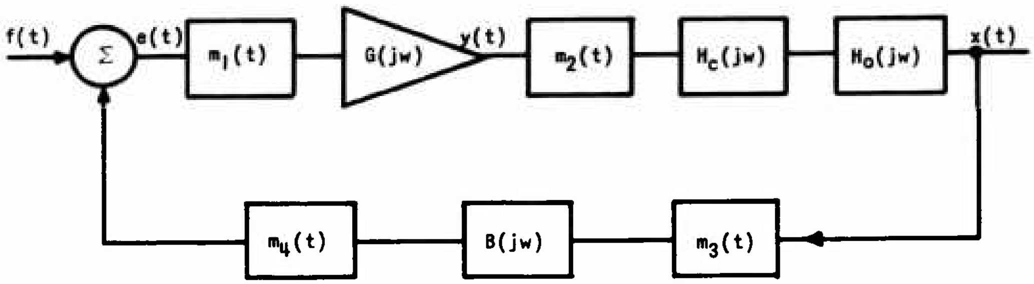


FIG. 30. A TIME-VARYING FEEDBACK SYSTEM.

except $H_c(jw)$ are specified in the above system, and that these quantities are Fourier transformable. $F(jw)$ denotes the Fourier transform of $f(t)$. It is desired to find the compensating filter $H_c(jw)$ such that the system will produce a specified response for the given input $f(t)$.

The system equations are obtained directly from Fig. 30.

$$\int_{-\infty}^{\infty} e(\tau)m_1(\tau)g(t - \tau)d\tau = y(t), \qquad g(t - \tau) \equiv 0 \quad \text{for} \quad \tau > t \,, \tag{4.30}$$

$$\int_{-\infty}^{\infty} y(\tau)m_2(\tau)h(t - \tau)d\tau = x(t), \qquad h(t - \tau) \equiv 0 \quad \text{for} \quad \tau > t \,, \tag{4.31}$$

$$e(t) = f(t) - m_4(t) \int_{-\infty}^{\infty} x(\tau)m_3(\tau)b(t - \tau)d\tau, \qquad b(t - \tau) \equiv 0 \quad \text{for} \quad \tau > t, \tag{4.32}$$

where $H(jw) = H_c(jw)H_o(jw)$, $e(t) \equiv 0$ for $t < 0$ but $m_i(t) \neq 0$, and $i = 1, 2, 3, 4$ for $t < 0$.

Solving these equations simultaneously, we obtain

$$H(jw) = \frac{X(jw)}{\{[E(jw)*M_1(jw)]G(jw)\}*M_2(jw)} \tag{4.33}$$

$$\text{and} \quad H_c(jw) = \frac{1}{H_o(jw)} H(jw) \,.$$

In order for $H_c(jw)$ to be physically realizable, it must satisy the Paley-Wiener criterion [Ref. 49]:

$$\int_{-\infty}^{\infty} \frac{\left|\log |H_c(jw)|\right|}{1 + w^2} \, dw < \infty . \tag{4.34}$$

If we set $jw = s$ in Eq. (4.33), we may use an alternate criterion for the physical realizability.

$$H_c(s) = \frac{X(s)}{\{[E(s)*M_1(s)]G(s)\}*M_2(s)} \frac{1}{H_o(s)} = \frac{N(s)}{D(s)} , \tag{4.35}$$

where $E(s) = M_4(s)*\{B(s)[M_3(s)*X(s)]\}$.

In order for $H_c(s)$ to be physically realizable, all the poles of $H_c(s)$ must lie in the left half of the s-plane.

We consider a special case in order to illustrate some of the problems involved in the synthesis of $H_c(s)$. If we make the following choice of components for the system shown in Fig. 30:

$$m_1(t) = m_2(t) = e^{j\Omega t} + e^{-j\Omega t}, \quad m_3(t) = m_4(t) = 1 , \quad \text{and} \quad B(s) = 1 ,$$

then the system represents a simple carrier feedback system. In this case, we obtain the following expression for $H_c(s)$ from Eqs. (4.30) and (4.31):

$$H_c(s) = \frac{1}{H_o(s)} \frac{X(s)}{E(s)[G(s+j\Omega) + G(s-j\Omega)] + E(s+j2\Omega)G(s+j\Omega) + E(s-j2\Omega)G(s-j\Omega)} \tag{4.36}$$

Since both $E(s)$ and $G(s)$ are rational functions of s, we set $E(s) = P_o(s)/Q_o(s)$ and $G(s) = -KN_o(s)/D_o(s)$, and substitute the resulting expressions into the above equation.

$$H_c(s) = -\frac{1}{K}\frac{X(s)}{H_o(s)}\frac{Q_oQ_1Q_{-1}D_1D_{-1}}{P_o(N_1D_{-1} + N_{-1}D_1)Q_2Q_{-2} + Q_o(P_2Q_{-2}N_1D_{-1} + P_{-2}Q_2N_{-1}D_1)}$$

$$= -\frac{1}{K}\frac{X(s)}{H_o(s)}\frac{N(s,\ j\Omega)}{D(s,\ j\Omega)}, \quad \text{where} \quad P_k = P(s + jk\Omega), \quad k = 0, \pm 1, \pm 2.$$

It is seen from the above equation that the order of $H_c(s)$ could be greater than the sum of the orders of $P_o(s)$, $Q_o(s)$, $N_o(s)$ and $D_o(s)$. Assuming that $X(s)/H_o(s)$ is analytic for Re $s > 0$, $D(s,\ j\Omega)$ must be a Hurwitz polynomial [Ref. 50], and $N(s,\ j\Omega)$ must be a polynomial of s with real coefficients, in order to satisfy the physical realizability criterion. Both $D(s,\ j\Omega)$ and $N(s,\ j\Omega)$ are in general polynomials of the form $\sum_{k=0}^{n} c_k s^k$ having real coefficients, c_k, as seen from Eq. (4.36). Some of the coefficients of $D(s,\ j\Omega)$, however, could be negative. It is, therefore, possible to realize $H_c(s)$ with time-invariant linear networks only in special cases.

The recent report by Weiss [Ref. 51] presents a reasonably complete theory of a special class of carrier frequency networks, but there are not yet theories which state precisely the constraints which would be necessary on $E(s)$ and $G(s)$ in Eq. (4.36) to insure the physical realizability of $H_c(s)$. There seem to be no published papers on the synthesis of $H_c(s)$ as shown in Eq. (4.35).

It is mathematically straightforward to obtain the "formal" solution of the compensation problem for the simple case considered above. However, the analytical complexity of formal solutions as seen from Eqs. (4.35) to (4.37) tends to make this formally elegant procedure rather impractical.

In consequence of the difficulties described above, a designer of carrier frequency control systems may find at present that the trial-and-error method of compensation with the aid of Floquet theory is more efficient and practical than the formal procedure based upon use of the integral equations.

V. SUGGESTIONS FOR FURTHER WORK

The work presented in the previous chapters may be extended in both theoretical and practical directions, as described below.

A. STABILITY OF LINEAR SYSTEMS WITH ALMOST PERIODIC COEFFICIENTS

The linear system with periodic coefficients is a subclass of the linear systems with almost periodic coefficients defined by

$$\dot{\underline{x}} = A(t)\underline{x} ,$$

where the elements $a_{ij}(t)$ of matrix $A(t)$ are almost periodic functions [Ref. 33, p. I-11, and Ref. 52]. In many cases, $A(t)$ may be written as a sum of periodic matrices, $A(t) = \sum_{i=1}^{k} A_i(t)$. The periods of matrices $A_i(t)$ are not necessarily rational multiples of each other. It is of interest to explore how stability of this system depends on the characteristic exponents of the subsystems defined by

$$\dot{\underline{y}} = A_i(t)\underline{y} , \quad i = 1, 2, \ldots k.$$

B. THE COMPENSATION AND DESIGN OF LINEAR SYSTEMS WITH PERIODIC PARAMETERS

We have shown a mathematical procedure for compensation of carrier frequency control systems. It is often impractical, however, to realize physically the compensating networks called for by the results of applying this procedure.

There seems to be a definite need for the development of simple, systematic procedures for the design of modulated control systems. Discovery of some clever approximate methods seems to be the key to the development of useful design procedures.

Normally, a design procedure may be divided into two parts, namely, approximation and synthesis. Since the form of impulse response (or the fundamental set of solutions) of a linear system with periodic parameters is known from the Floquet theory, one can specify the desired

response characteristics in terms of the impulse response.

In order to realize a specified impulse response $g(t)$ by a linear system with periodic parameters, it is necessary in general to approximate $g(t)$ by a realizable impulse response of the form $h(t) = \sum_{i=1}^{k} c_i p_i(t) e^{\beta_i t}$, where c_i are constants and $p_i(t)$ are periodic functions. Having determined c_i, $p_i(t)$ and β_i according to a chosen approximation criterion, then one has to develop the synthesis procedure for physical realization of the approximate impulse response $h(t)$.

APPENDIX A. PROOF OF THE EXTENDED FLOQUET THEORY

This appendix presents a proof for the extended Floquet theory developed in Chapter II.

Referring to Chapter II, the solution of Eq. (2.66) is

$$\underline{x}(t) = \Phi(t)\underline{x}(0) + \int_0^t \Phi(t)\Phi^{-1}(\tau)\underline{b}(\tau)d\tau \ . \tag{A.1}$$

Now we set $t = k + q$; $k < q < k + t_1$; then

$$\underline{x}(k + q) = \Phi(k + q)\underline{x}(0) + \int_0^{k+q} \Phi(k + q)\Phi^{-1}(\tau)\underline{b}(\tau)d\tau \ . \tag{A.2}$$

Since $\Phi(k + 1 + q) = \Phi(k + q)C$ from Eq. (2.61), $\Phi(k + q) = \Phi(q)C^k$, and the integral may be written as

$$\begin{aligned}\Sigma(k) &= \Phi(k + q)\left[\int_0^1 \Phi^{-1}(\tau)\underline{b}(\tau)d\tau + \int_1^2 \Phi^{-1}(\tau)\underline{b}(\tau)d\tau + \ldots\right. \\ &\quad \left. + \int_{k-1}^{k} \Phi^{-1}(\tau)\underline{b}(\tau)d\tau + \int_k^{k+q} \Phi^{-1}(\tau)\underline{b}(\tau)d\tau\right] \\ &= \Phi(q)C^k \int_0^1 [\Phi^{-1}(\tau)\underline{b}(\tau) + C^{-1}\Phi^{-1}(\tau)\underline{b}(\tau + 1) + \ldots \\ &\quad + C^{-(k-1)}\Phi^{-1}(\tau)\underline{b}(\tau + k - 1) + C^{-k}\Phi^{-1}(\tau)\underline{b}(k + \tau)]d\tau\end{aligned} \tag{A.3}$$

from which:

$$\begin{aligned}\Sigma(k) &= \Phi(q) \int_0^1 [C^k\Phi^{-1}(\tau)\underline{b}(\tau) + C^{k-1}\Phi^{-1}(\tau)\underline{b}(\tau + 1) + \ldots \\ &\quad + \Phi^{-1}(\tau)\underline{b}(\tau + k)]d\tau \ .\end{aligned}$$

Rewriting Eq. (A.2) as

$$x(k + q) = C^k\Phi(q)\underline{x}(0) + \Sigma(k) \ , \tag{A.4}$$

and taking the norms of both sides, we have

$$\|\underline{x}(k + q)\| \leq \|C^k\| \ \|\Phi(q)\underline{x}(0)\| + \|\Sigma(k)\| \ . \tag{A.5}$$

Since $q < 1$, $\|\Phi(q)\| \leq a_1$ and we set $\|\underline{x}(0)\| \leq x_o$, $\|\Phi^{-1}(\tau)\| \leq a_2$, $0 \leq \tau \leq 1$, where a_1, a_2 and x_o are finite, positive, real numbers. Now we examine the two terms of Eq. (A.5) separately.

1. In order to find an upper bound of $\|C^k\|$ for the general case, we assume that the maximum characteristic roots of matrix C have multiplicity α. Since an arbitrary matrix can be transformed into the Jordan normal form, it is always possible to set

$$C = U^{-1}JU \ , \tag{A.6}$$

where J is a matrix in Jordan form. Then it follows at once that

$$C^k = U^{-1}J^kU \ ; \tag{A.7}$$

and if $k > \alpha$, then

$$J^k = \begin{bmatrix} z_1^k & kz_1^k & \cdots & \dfrac{k! \ z_1^{k-\alpha}}{\alpha!(k-\alpha)!} \\ 0 & z_1^k & \ddots & \\ 0 & & & \ddots \end{bmatrix} .$$

Assuming that z_1 is the maximum characteristic root but $|z_1| < 1$ by hypothesis, we have

$$\|J^k\| \leq \frac{k!}{\alpha!(k - \alpha)!} |z_1|^{k-\alpha} < k^\alpha |z_1|^{k-\alpha} \quad \text{for} \quad k > \alpha \ , \tag{A.8}$$

or

$$\|J^k\| \leq N(\alpha) \quad \text{for} \quad k < \alpha \ ,$$

and

$$\|C^k\| \leq \|U^{-1}\| \, \|J\| \, \|U\| < u_o k^{\alpha} |z_1|^{k-\alpha}, \tag{A.9}$$

where $\|U^{-1}\| \, \|U\| \leq u_o$, and $N(\alpha)$ is a bounded positive constant.

2. An upper bound of $\|\Sigma(k)\|$ is readily found to be

$$\|\Sigma(k)\| \leq \|\Phi(q)\| \, [\|C^k\| + \ldots + \|C^{\alpha}\| + \alpha N(\alpha)] a_2 b_o$$

$$< a_1 a_2 b_o [k^{\alpha} |z_1|^{k-\alpha} + \ldots + \alpha^{\alpha} + \alpha N(\alpha)] u_o$$

$$= M[k^{\alpha} |z_1|^{k-1} + \ldots + \alpha^{\alpha} + \alpha N(\alpha)] , \tag{A.10}$$

where $M = a_1 a_2 b_o u_o$. Consequently we obtain the inequality

$$\|\underline{x}(k + q)\| < a_1 u_o x_o k^{\alpha} |z_1|^{k-\alpha} + M[k^{\alpha} |z_1|^{k-\alpha} + \ldots + \alpha^{\alpha} + \alpha N(\alpha)] . \tag{A.11}$$

Since $|z_1| < 1$ by assumption, we can set

$$z_1 = e^{-\beta}, \quad \beta > 0 . \tag{A.12}$$

Then $k^{\alpha} |z_1|^{k-\alpha} = k^{\alpha} e^{-\beta(k-\alpha)}$, and this becomes arbitrarily small as $k \to +\infty$.

It is clear from this argument that

$$\lim_{k \to \infty} \|\underline{x}(k + q)\| < M\alpha[\alpha^{\alpha-1} + N(\alpha)] \tag{A.13}$$

and this limit is bounded.

Therefore the hypothesis that the magnitude of the maximum characteristic root, z_1, is less than one (i.e., all roots lie within the unit circle) is sufficient to guarantee stability of the solution of the linear system with piece-wise continuous periodic coefficients described by Eq. (2.69); that $|z_1| < 1$ is also necessary is evident from the preceding argument. This statement is equivalent to the theorem of Chapter II.

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